A Categorical Framework for Higher Order Physical Theories

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We develop a mathematical framework for higher order physical theories, in which the dynamics of physical systems can be acted upon by higher-order physical processes called supermaps. We start by defining the notion of Super-Monoidal Category a constraint imposed on the notion of a $\mathcal{V}$-Monoidal Category, whose morphisms represents supermaps acting on a base category of physical processes. We proceed to define the notion of a theory which permits the static manipulations of all of its dynamic processes, showing that when equipped with a particular well motivated isomorphism they in fact co-incide with closed monoidal categories. We then use the framework to present a theorem on the inevitability of closed monoidal structure for theories which contain infinite towers of super-physical theories, each layer being a theory of manipulations of the processes within the layer beneath it. Super-Monoidal Categories, being an axiomatisation of the basic features of theories of supermaps provide a broad framework for the study of novel causal structures in quantum theory, and, more broadly, provide a paradigm of physical theory where static and dynamical features are treated in a unified way.

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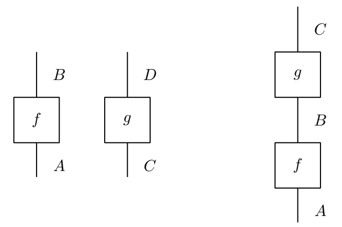
1 Introduction

Traditionally, physical theories have been concerned with the laws governing the evolution of certain physical systems, such as particles or fields. In the ontology of a theory, the physical systems are regarded as fundamental objects, while their evolution is regarded as a tool for predicting elations among objects in different regions of space and time. Over the past decade, a series of works in quantum information theory started exploring the idea that processes themselves could be regarded as objects, which can be acted upon by a kind of higher order physical transformations, known as quantum supermaps [1–7]. Quantum supermaps have found a wide range of applications to quantum information and computation [8–23], and to the study of new types of causal structures arising in quantum mechanics [3, 24–26]. In addition, higher order transformations provide a broad framework for general physical theories with dynamical causal structure, and, eventually, are expected contribute to the formulation of a complete theory of quantum gravity, as originally suggested by Hardy [27]. Complementary to this research direction is the development of programming languages which permit higher order types whilst retaining compatibility with quantum theory (by forbidding cloning [28], the signature of the cartesian monoidal structure underlying the standard lambda calculus), such as *linear* or *quantum* lambda calculi [29–34].

A compositional foundation for the study of physical theories, including quantum and classical theory, is provided by the process theory framework [35]. The framework is built on the notion of symmetric monoidal category, which captures the basic structures required in a broad class of physical theories. Such structures include a notion of system, a notion of processes between systems, and, crucially, a notion of the sequential and parallel composition of processes, diagrammatically represented as



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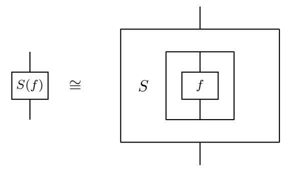
The tensor unit I represents the *trivial system*, and, for every object A, the processes of type $I \to A$ are viewed as *states* of system A. The process theoretic treatment has contributed new insights and intuitions to quantum foundations and the general structure of physical theories [36–53]. In addition to the above formal circuit based representation for quantum processes, supermaps are often informally represented by boxes with open holes, into which processes may be inserted:



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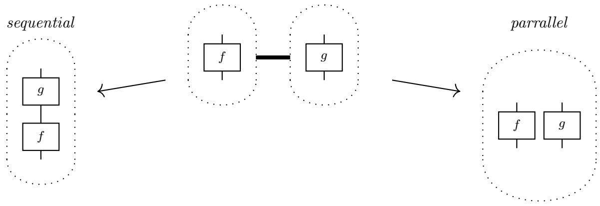
In a recent work [54], a process theoretic framework for supermaps was developed for the purpose of providing a categorical language for causal structures. In this framework, causal structures are represented by the objects of a $*$-autonomous category of higher order processes $\mathbf{Caus}(\mathcal{C})$ built on from a "pre-causal" category $\mathcal{C}$. This construction revealed deep relations between $*$-autonomy and the structure of higher order transformations in quantum theory, in particular producing a convenient type system for reasoning about causal structures. Building on this work, there is a sense that the notion of a raw-material pre-causal (and so compact closed) category will be too restrictive a requirement in the study of infinite dimensional systems such as those encountered in quantum field theory, and ultimately quantum theories of gravity. Furthermore the authors anticipate that it would be fruitful to pin down the notion of a higher order theory as a mathematical structure in its own right, independently of the study of causality, and independently of the notion of a raw material category from which a theory might be constructed. Such a framework would potentially contribute to the initiation of a new research direction, the study of the effect of imposing physical principles onto abstract higher order theories.

The first contribution of this paper is to provide a categorical notion for super-physics, via the definition of a *Super-Monoidal Category*, the world in which we claim such a higher order theory must live. The objects in a Super-Monoidal Category correspond to types of physical processes, and the morphisms represent the possible supermaps transforming processes into processes. The key principle of a Super-Monoidal theory, is that its should be possible to *implement* the external structural features of a standard symmetric monoidal category (process theory), concretely the sequential and parallel composition structure which might *occur* in a standard theory of physics, become manipulations that may be *implemented* by higher order transformations of those processes. In short a super-monoidal theory is one in which black box processes of an underlying theory may be plugged together by higher order transformations, either in sequence or in parallel.



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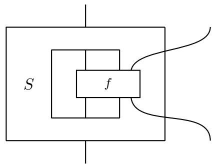
The second contribution of this paper is a categorical phrasing of the notion of a *completely parallel* theory, a *cp-Super Monoidal r-Category*: One in which every super-map may be applied to part of any bipartite process.



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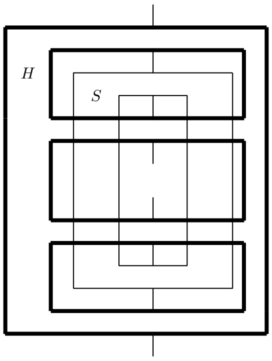
After laying the ground work we note that the notion of a theory of supermaps may be iterated, to provide a formalisation of the notion of a theory of super-super maps and so on.



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We then consider theories which are conveniently infinitely iterated as a consequence of being *internal* or *self-contained*, precisely those theories with the property that they are their own theory of higher order transformations. We observe that this leads to precisely a characterisation of closed monoidal categories. The characterisation is entirely operationally motivated by two key principles which ensure a physical theory be "complete", meaning that

- The theory permits the manipulation of all higher order transformations
- There is an equivalence between having access to a generic system type A and the higher order type $[I, A]$ associated to states on A - the processes from the monoidal unit I to type A.

We call such categories *Linked Super-Monoidal r-Categories* and prove that they are exactly closed symmetric monoidal categories.

Theorem 1. *A category $\mathcal{C}$ is a Linked Super-Monoidal r-Category if and only if it is a closed symmetric monoidal category.*

We finish by relaxing the above approach to deriving closed monoidal structure for higher order theories, considering the notion of a tower of theories-each a theory of supermaps for the theory bellow it. We provide a suitable notion of a tower of Super-Monoidal Categories, with the higher levels of the hierarchy representing physical transformations on the lower levels. Operationally, one can associate higher levels to more powerful agents, who have the ability to manipulate more and more complex physical objects. The notion of a Merger of such a hierarchy is introduced as a candidate for the notion of a theory in which the inhabitants are all-powerful, with the capability to manipulate objects of any level in a hierarchy of super-theories. For this definition, we formally require that a hierarchy of Super-Monoidal Categories be embedded into a single symmetric monoidal category, and that the different levels of the hierarchy are *linked* so that it is possible to transition between the layers of the hierarchy, formally this linking is modelled by the existence of an isomorphism $A \simeq [I, A]$ between each system type A and the higher order type $[I, A]$ representing the space of states on (elements of) A. We use the above structure to propose a fundamental object of study in higher order physics,

Definition 1. *A Merger for a coherent sequence of Super-Monoidal categories $\mathcal{C}^{(i)}$ is a 2-Cone $(\mathcal{F}_i : \mathcal{C}_i \longrightarrow \mathcal{C})$ over the diagram*

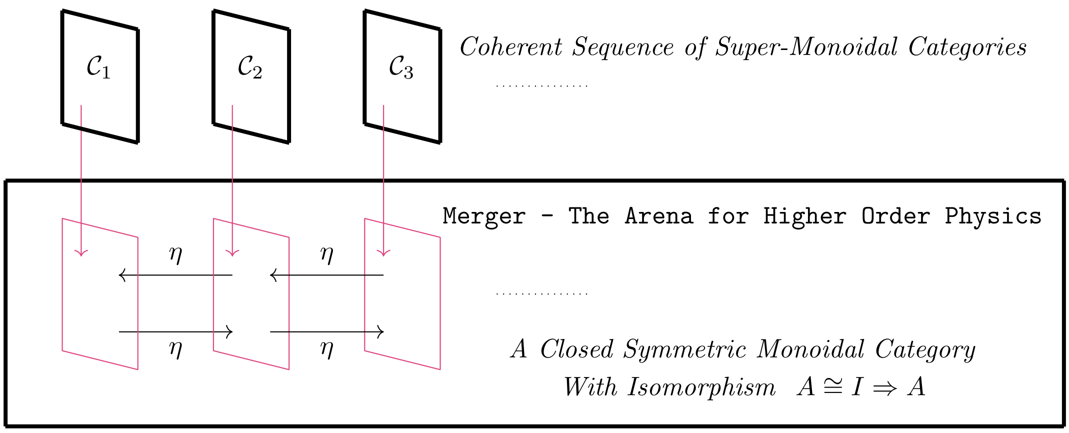
$$\dots \qquad \mathcal{C}_{i-1} \xrightarrow{\mathcal{R}^{i}_{i-1}(-)} \mathcal{C}_i \xrightarrow{\mathcal{R}^{i+1}_{i}(-)} \mathcal{C}_{i+1} \qquad \dots$$

in **ffSymCat***: the 2-Category of Symmetric monoidal categories, symmetric monoidal functors, and symmetric monoidal natural transformations, such that*

- *The canonical functor $\coprod_i \mathcal{F}_i : \coprod_i \mathcal{C}_i \to \mathcal{C}$ from the co-product $\coprod_i \mathcal{C}_i$ in* **Cat** *is essentially surjective*

The final technical contribution of this manuscript is to demonstrate that every such higher order theory possesses a simple, convenient, and familiar categorical property, that of being closed.

Theorem 2. *The apex $\mathcal{C}$ of any Merger of infinite order is a Closed Symmetric Monoidal Category.*

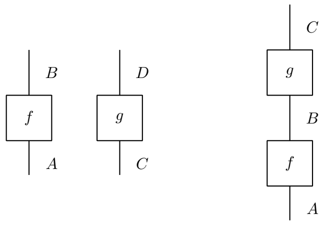


This theorem derives the notion of closed-ness without reference to any physical feature of a theory *except* the condition that it be an infinite tower of higher order physical theories. Closed monoidal structure is the key notion for linear lambda calculi [55, 56] and so our main theorem entails that every complete higher order theory of physics over a base physical theory $\mathcal{C}_1$ will permit a notion of $\mathcal{C}_1$-Lambda Calculus, in turn this connection along with the results of [54] may aid the cross pollination of results between the development of quantum lambda calculi [29–32] and the study of causal structure and higher order physics from the perspective of GPTs [57] and CPTs [58].

The connections between new [59] and old [60] categorical structures uncovered result in a convenient end product for the study of higher order physics, a closed monoidal category. A conceptual takeaway from these results is that the possibility for agents to plug processes together in sequence and in parallel, along with a linking between lower and higher order levels of a theory, are sufficient to operationally justify the convenient mathematical structure of the possibility to curry any process. Just as with the notion of a compact closed category, this categorical notion can easily be exported to the process theoretic framework for physics, now with strong motivation to do so. With a sound framework in place, the imposition of physical principles on top of higher order structure, the development higher order extensions of infinite dimensional quantum, field-theoretic, and post-quantum process theories [61–67], the characterisation of the notion of higher order measurement [38, 68], the exploration of the consequences for higher order resource theories [69, 70], the relation to open diagrams in terms of the co-end calculus [71, 72], the intersection with probabilistic [58], inferential theories [73], and the extension to operational theories with time symmetry [74, 75] present a non-exhaustive list of research avenues that could in principle stem from this framework.

2 Notation and basic definitions

Here we flag a few standard categorical notions that will be often used in our framework. The formal definition of each structure in *italics* is provided in the appendix for reference. We will use the abbreviation SMC for *symmetric monoidal category*, the definition of which may be found in [60], and we will often represent morphisms with string diagrams [76]. A morphism $f : A \to B$ is represented by a box with input wire A and outut wire B. The $\otimes$ composition $f \otimes g$ is written by placing f next to g.



(6)

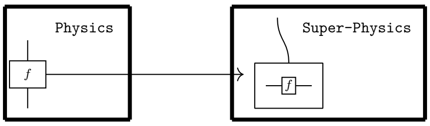
The unit I is not written, interpreted as representing only empty space. Similarly associativity of sequential composition and associativity of parallel composition up to natural isomorphism are absorbed into the graphical language, neither being explicitly written. The categorical notion of one monoidal category living inside another is that of a *monoidal functor*. In general, there may be more than one way of representing a monoidal category $\mathcal{C}$ inside another monoidal category $\mathcal{D}$. The notion of equivalence between two representations is that of a *monoidal natural isomorphism*. A key notion for the description of supermaps will be that of an *enriched category* [77], which in essence is the lifting of the notion of category so that the composition of processes is itself a morphism in a category generalising the notion of a composition function between sets of morphisms. A key notion for the description of a complete higher order physical theory will be that of a *closed symmetric monoidal category* (closed SMC) [55]: the structure which captures the notion of currying with respect to the monoidal product $\otimes$ of an SMC.

3 Higher Order Physical Theories: Super Monoidal r-Categories

The first contribution of this paper is to provide a categorical framework for supermaps. In this section we provide the mathematical definitions, after presenting their basic operational motivation.

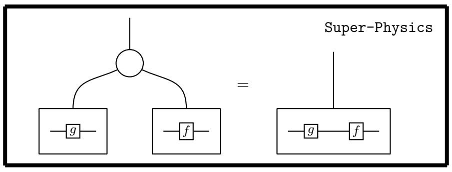
3.1 The Most Basic Manipulations of Processes

A physical theory of supermaps is a second-order theory, built on top of a first-order physical theory. The states in the second-order theory are identified with processes in the underlying first-order theory, as in the following illustration:



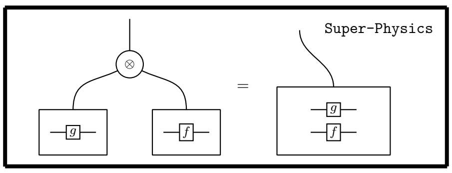
(7)

With the objects to be manipulated in place, the second question is that of the most fundamental super-maps, manipulations of processes that should exist in any higher order theory. It should at the very least be possible to plug those processes together in sequence, wiring the output of one into the input of the other.



(8)

It should furthermore be possible to combine any two processes by simply placing them next to each-other.



(9)

Formally, this means that a theory of super-maps should be at the very least an extension of the notion of enrichment [78] to include a parallel composition morphism.

3.1.1 The Categorical Language of Basic Manipulations

We first present the data of a $\mathcal{V}$-Symmetric Monoidal Category $\mathcal{C}$ in a non-standard way, with the aim of making explicit how the categorical notion genuinely models the desired physical scenario. After-which we then rephrase the notion in terms more familiar to standard enriched category theory.

Definition 2. *A $\mathcal{V}$-Symmetric Monoidal r-Category $\mathcal{C}$ is a symmetric monoidal category $(\mathcal{V}, \bullet, \otimes_{\mathcal{V}})$ and a symmetric monoidal category $(\mathcal{C}, \circ, \otimes_{\mathcal{C}})$ such that for every pair of objects A, B of $\mathcal{C}$ there exists an object $[A, B]$ of $\mathcal{V}$ satisfying $hom_{\mathcal{C}}(A, B) = \mathcal{V}(I, [A, B])$ and such that there exist families of morphisms (indexed by the objects of $\mathcal{C}$)*

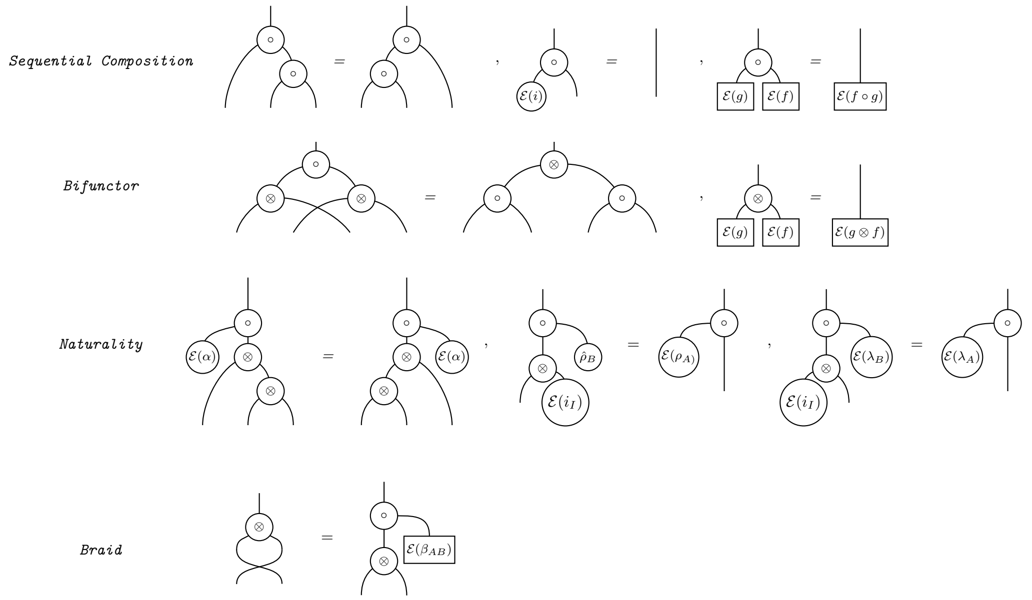
- $\bigcirc : [A, B] \otimes [B, C] \to [A, C]$ *and*

- $\boxtimes : [A, A'] \otimes [B, B'] \to [A \otimes B, A' \otimes B']$

which along with the embedding of the canonical morphisms of $\mathcal{C}$:

- $\mathcal{E}(i_A) : I_{\mathcal{V}} \to [A, A]$
- $\mathcal{E}(\lambda_A) : I_{\mathcal{V}} \to [I \otimes A, A]$
- $\mathcal{E}(\rho_A) : I_{\mathcal{V}} \to [A \otimes I, A]$
- $\mathcal{E}(\alpha_{ABC}) : I_{\mathcal{V}} \to [(A \otimes B) \otimes C, A \otimes (B \otimes C))]$
- $\mathcal{E}(\beta_{AB}) : I_{\mathcal{V}} \to [A \otimes B, B \otimes A]$

Satisfy the following conditions.



(10)

The structural morphisms of a $\mathcal{V}$-Symmetric Monoidal Category $\mathcal{C}$ represent the basic manipulations that should be performable in a higher order theory, sequential composition, parallel composition, and the re-ordering of wires, alongside the existence of additional coherence morphisms λ, ρ, α which represent nothing other than the book-keeping of empty space. The standard definition of a $\mathcal{V}$-Symmetric Monoidal Category is the following

Definition 3. *A $\mathcal{V}$-Symmetric Monoidal Category $\mathcal{C}$ is a $\mathcal{V}$-Enriched Category $\mathcal{C}$ equipped with a $\mathcal{V}$-Functor $\boxtimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ an object I and a family of $\mathcal{V}$-natural isomorphisms*

- $\alpha : (A \boxtimes B) \boxtimes C) \to (A \boxtimes (B \boxtimes C))$
- $\lambda : I \boxtimes A \to A$
- $\rho : A \boxtimes I \to A$
- $B : A \boxtimes B \to B \boxtimes A$

The underlying category will be notated by $\mathcal{C}_0$. The explicit definition of a $\mathcal{V}$-Symmetric Monoidal r-Category $\mathcal{C}$ can be packaged in standard category theoretic parlance in the following way.

Lemma 1. *The data required to define either of:*

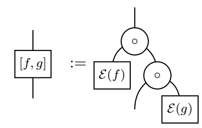
- *A $\mathcal{V}$-Symmetric Monoidal r-Category $\mathcal{P}$*
- *A $\mathcal{V}$-Symmetric Monoidal Category $\mathcal{C}$ and a symmetric monoidal category $\mathcal{P}$ equipped with a strict full faithful monoidal and injective and surjective on objects functor $r : \mathcal{P} \longrightarrow \mathcal{C}_0$*

are equivalent.

The proof of the above is trivial, and puts onto a formal setting the sense in which a $\mathcal{V}$-Symmetric Monoidal r-Category $\mathcal{C}$ is simply a standard monoidally enriched category equipped with a re-labelling. It will serve us well during our analyses to think in terms of the more compact notion a $\mathcal{V}$-Symmetric Monoidal r-Category $\mathcal{C}$, ignoring any intermediate notion of a standard monoidally enriched category.

3.1.2 Basic Properties of Any $\mathcal{V}$-Symmetric Monoidal Categories

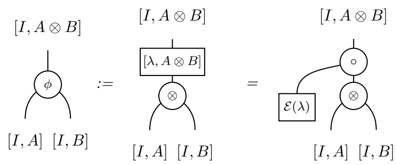
For any $\mathcal{V}$-Symmetric Monoidal Category $\mathcal{C}$ the assignment $[-,-]$ extends to a hom functor $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$ defined by



(11)

The existence of the parallel composition morphism entails that the induced functor $[I,-]$ be a braided monoidal functor

Theorem 3. *The data $([I,-], \phi, \mathcal{E}(i_I))$ with the natural transformation $\phi : [I,-] \otimes [I,-] \longrightarrow [I, - \otimes -]$ given by*



(12)

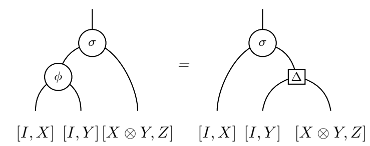
defines a braided lax monoidal functor

$$[I,-] : \mathcal{C} \to \mathcal{V}$$

Proof. A standard result of enriched category theory, given in the appendix for completeness. □

Physically, the natural transformation ϕ is interpreted as a process that takes two states and interprets them as a bipartite state, morally its meaning is identical to that of the parallel composition process $\otimes$. A key conceptual and technically critical feature of $\mathcal{V}$-Symmetric Monoidal Categories is the existence of a higher order process representing partial use of a lower order process (the usage of one of its inputs).

Lemma 2. *For any $\mathcal{V}$ Symmetric Monoidal r-category $\mathcal{C}$ there exists an operation Δ, named* partial insertion*, which satisfies the following condition:*



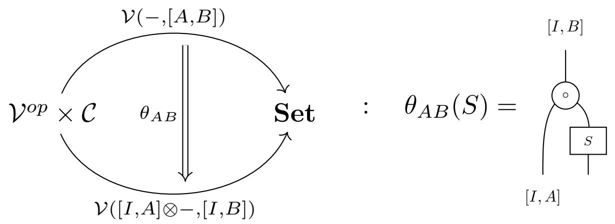
(13)

Proof. Given in the Appendix ☐

The existence of a partial insertion is a pre-cursor to the notion of currying in a closed monoidal category. An agent in the super-theory can take any bipartite process, and use only one of its inputs, leaving the rest available for use at a later date.

Finally, for any supermap, there exists a usage transformation, which takes the output process of a supermap and applies it to the specified input state space.

Theorem 4. *There exists a natural transformation defined by*



Termed the `usage` *transformation.*

Proof. Immediate by associativity of ◯. ☐

The usage transformations can furthermore be viewed as the components of a $[\mathcal{V}^{op}, \mathbf{Set}]$-Functor

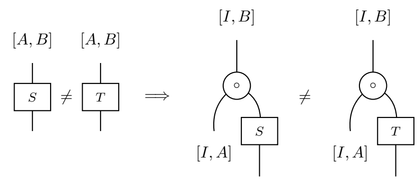
$$\theta : \mathcal{V}(-,[A,B]) \longrightarrow \mathcal{V}([I,A] \otimes -, [I,B])$$

Indeed each component θ_{AB} is a natural transformation, a morphism in the functor category, the functor category can be viewed as monoidal with respect to the day tensor product, and with respect to this tensor product the above hom object assignments can be assigned famillies of composition morphisms by (?), and the morphisms θ_{AB} can be shown to obey the standard enriched functor conditions with respect to those families.

3.2 Well-Behavior of Manipulations: Super-Monoidal Categories

On top of a $\mathcal{V}$-Symmetric Monoidal r-Category $\mathcal{C}$, there are additional conditions that should be expected of a theory of higher order processes $\mathcal{V}$ on the processes of a theory $\mathcal{C}$.

- Since the interpretation of a higher order process $S : X \to [A,B]$ is that it produces a process $f : A \to B$ of the lower order theory, two higher order processes $S, T : X \to [A,B]$ should *only* be distinguishable if they are distinguishable when their outputs are applied to the space of states on A. Stated formally For all I, A, B the composition process $\bigcirc_{IAB}$ ought to satisfy



(14)

Categorically this condition is precisely the requirement that the natural transformation θ be a monomorphism in the functor category $\mathbf{Cat}(\mathcal{V}^{op} \otimes \mathcal{C}, \mathbf{Set})$, since the domain of all functors in $\mathbf{Cat}(\mathcal{V}^{op} \otimes \mathcal{C}, \mathbf{Set})$ is $\mathbf{Set}$ the monomorphisms between them are simply the natural transformations for which each component is a monomorphism (injection) in **Set**. This has the additional consequence of entailing that the functor $[I, -]$ be faithful. Furthermore when θ is viewed as a $[\mathcal{V}^{op}, \mathbf{Set}]$-Functor the condition that each component θ_{AB} be a monomorphism is equivalent to the requirement that θ be full and faithul as an enriched functor.

- One should expect that all correlated states on the joint system $A \otimes B$ in a standard physical theory should also exist as correlations on the joint space $[I, A] \otimes [I, B]$ of states on A and states on B. In other words one should expect a natural isomorphism $[I, A] \otimes [I, B] \cong [I, A \otimes B]$. Categorically this is the requirement of strength for the monoidal functor $[I, -]$. A note worth making is that for non-cartesian monoidal physical theories, this immediately forbids any higher order theory $\mathcal{V}$ over $\mathcal{C}$ from being cartesian monoidal. In particular the trivial **Set**-Symmetric Monoidal r-Category $\mathcal{C}$ can only be a sensible higher order theory when $\mathcal{C}$ is cartesian I.E it has no non-trivial joint states.

- The allowed processes from the states $[I, A]$ of A to the states $[I, B]$ of B should simply be the allowed processes from A to B, this is the requirement that the monoidal functor $[I, -]$ be full.

Definition 4 (**A Primitive Theory of Supermaps**). *A* $\mathcal{V}$-Super-Monoidal r-Category $\mathcal{C}$ *is a* $\mathcal{V}$*-Symmetric Monoidal category* $\mathcal{C}$ *such that*

- *The usage transformation* $\theta : \mathcal{V}(-, [A, -]) \Longrightarrow \mathcal{V}([I, A] \otimes -, [I, -])$ *is a monomorphism in the functor category* $\mathbf{Cat}(\mathcal{V}^{op} \otimes \mathcal{C}, \mathbf{Set})$.

- *The faithul monoidal functor* $[I, -]$ *is full and strong.*

Example 1 (Quantum Supermaps - Low Order Types in Higher Order Causal Categories)**.** *In any Higher Order Causal Category* $\mathbf{Caus}[\mathcal{C}]$*, the first order types* $(A, \{d\})$ *represent the sub-category* $\mathbf{Caus}_1[\mathcal{C}]$ *causal processes, in the case of quantum theory they represent CPTP maps. One can then construct a* $\mathbf{Caus}[\mathcal{C}]$ *Super-Monoidal r-Category* $\mathbf{Caus}_1[\mathcal{C}]$ *using the fact that* $\mathbf{Caus}[\mathcal{C}]$ *is closed monoidal to make the assignment* $\mathcal{E}(f) = \hat{f}$ *where* $\hat{f}$ *is the adjunct to* f*. Similarly sequential and parallel composition morphisms can be constructed in the usual way under adjunction to circuits of evaluation morphisms. All coherence conditions follow from coherence conditions with respect to the closed monoidal structure of* $\mathbf{Caus}[\mathcal{C}]$ *along with the universal property of evaluations.*

Given two theories of super-physics, that is, a $\mathcal{V}_1$*-Super-Monoidal r-Category* $\mathcal{C}_1$ and a $\mathcal{V}_2$*-Super-Monoidal r-Category* $\mathcal{C}_2$ a notion of homomorphism between the two can be defined, a functor from $\mathcal{C}_1$ to $\mathcal{C}_2$ a functor from $\mathcal{V}_1$ to $\mathcal{V}_2$ which preserves the manipulation structure:

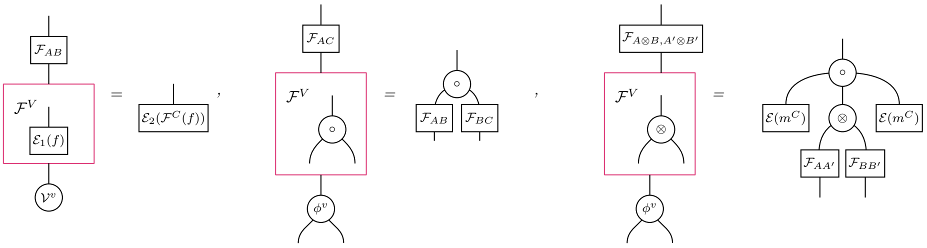
Definition 5. *A super monoidal r-functor* $\mathcal{F}$ *between a* $\mathcal{V}_1$*-Super Monoidal r-Category* $\mathcal{C}_1$ *and a* $\mathcal{V}_2$*-Super Monoidal r-Category* $\mathcal{C}_2$ *is*

- *A monoidal functor* $(\mathcal{F}^C, m^C, v^C) : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$

- *A monoidal functor* $(\mathcal{F}^V, m^V, v^V) : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$

- *For each* $A, B \in \mathcal{C}$ *a morphism* $\mathcal{F}_{AB} : \mathcal{F}^V([A, B]_1) \to [\mathcal{F}^C(A), \mathcal{F}^C(B)]_2$

such that $\mathcal{F}_{AB}$ *satisfies*



(15)

A $\mathcal{V}$*-super monoidal r-functor* $\mathcal{F}$ *is furthermore termed strong if* $\mathcal{F}^C$ *and* $\mathcal{F}^V$ *are strong, and termed full and faithul if* $\mathcal{F}^C$*,* $\mathcal{F}^V$ *are full and faithul and every* $\mathcal{F}_{AB}$ *is an isomorphism.*

The definition of a super-monoidal functor is analogous to the notion of a $\mathcal{V}$-Monoidal functor [59] up to the accounting for the additional structure of the underlying category $\mathcal{C}$ and including the possibility that the category $\mathcal{V}$ also be subject to change. Super monoidal r-functors do indeed define a notion of morphism between super monoidal r-categories, they permit a notion of composition and so form a category.

Theorem 5. *Super Monoidal r-Categories and the Super Monoidal r-Functors between them define a category* **SupMonrCat***.*

Proof. The composition $\mathcal{F} \circ \mathcal{G}$ of a super monoidal r-functor $\mathcal{F}$ and a super monoidal r-functor $\mathcal{G}$ is given by

- $(\mathcal{F} \circ \mathcal{G})^C := \mathcal{F}^C \circ \mathcal{G}^C$
- $(\mathcal{F} \circ \mathcal{G})^V := \mathcal{F}^V \circ \mathcal{G}^V$
- $(\mathcal{F} \circ \mathcal{G})_{AB} := \mathcal{F}_{\mathcal{G}^C(A)\mathcal{G}^C(B)} \circ \mathcal{F}^V(\mathcal{G}_{AB})$

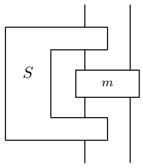
A full proof that this composition is associative and indeed defines a super monoidal r-functor $\mathcal{F}$ is given in the appendix. □

As a fruit of providing the notion of functor between higher order theories we get for free the categorical invariant notion of a sub-theory as a faithful version of such a functor.

Example 2. *The theory of non-signalling LOCCP supermaps is a higher order sub-theory of the theory of quantum supermaps in the sense that it embeds into the* $\mathcal{V}$ *Super Monoidal r-Category* $\mathcal{C}$ *of quantum supermaps via a faithful super monoidal r-functor* $\mathcal{F} : (\mathcal{V}_{\mathbf{LOCC}}, \mathcal{C}_{\mathbf{LOCC}}) \longrightarrow (\mathcal{V}, \mathcal{C})$ *given on objects by* $\mathcal{F}^C(X) := \mathcal{E}^C(X)$ *on morphisms by* $\mathcal{F}^C(f) = f$ *similarly for* $\mathcal{F}^V$*. The components* $\mathcal{F}_{A,B}$ *are all identities.*

4 Complete Parallelism in Higher Order Theories: cp-Super Monoidal r-Categories

Having addressed the treatment of processes as independent objects that may be arranged and plugged together we propose a model for the final notion of a theory of supermaps, the condition that any supermap, when applied to part of *any* bipartite process, may return a new bipartite process.

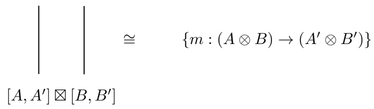


(16)

To include a model of this feature we require that there be a second notion of parallel composition $\boxtimes$ which treats processes as already glued together. We ask for a (natural) isomorphism of the form

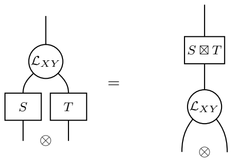
$$[A, A'] \boxtimes [B, B'] \cong [A \otimes B, A' \otimes B']$$

so that $\boxtimes$ represents all bipartite channels as opposed to the independently manipulable ones.



(17)

We also ask that there should be a relationship between the two tensor products, a sense in which for any pair of supermaps $S : [A, A'] \to [B, B']$ and $T : [C, C'] \to [D, D']$ the two tensor products $S \otimes T$ and $S \boxtimes T$ should be the same. Extended more generally to all objects, we expect to have some embedding $\mathcal{L}_{XY} : X \otimes Y \to X \boxtimes Y$ of the space $X \otimes Y$ which treats X and Y as more freely manipulable, and the system $X \boxtimes Y$ which more glues the systems X and Y together. This embedding should witness the similarity between $S \otimes T$ and $S \boxtimes T$.



(18)

This embedding notion is captured by a lax monoidal functor $\mathcal{L} : (\mathcal{V}, \boxtimes, I) \to (\mathcal{V}, \otimes, I)$, where the lax-ness captures the notion that $X \otimes Y$ lives inside of $X \boxtimes Y$. We call super-monoidal categories which support this second notion of parallel composition as completely-parallel. The term is specifically chosen to have its initials "cp" co-inside with the usual term given to the notion of complete parallelism in quantum information theory, the notion of "complete positivity".

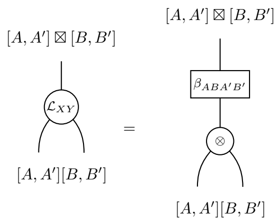
Definition 6 ($\mathcal{V}$ cp-Super Monoidal Category $\mathcal{C}$)**.** *A $\mathcal{V}$ cp-Super Monoidal Category $\mathcal{C}$ is a $(\mathcal{V}, \otimes_{\mathcal{V}},, I)$ Super Monoidal Category $(\mathcal{C}, \otimes_C, I_C)$ equipped with*

- *An additional monoidal structure $(\mathcal{V}, \boxtimes_{\mathcal{V}}, I)$*
- *A natural isomorphism $\beta : [A \otimes_{\mathcal{C}} B, A' \otimes_{\mathcal{C}} B'] \cong [A, A'] \boxtimes_{\mathcal{V}} [B, B']$*
- *A lax monoidal functor $\mathcal{L} : (\mathcal{V}, \boxtimes_{\mathcal{V}}, I) \longrightarrow (\mathcal{V}, \otimes_{\mathcal{V}}, I)$*

such that the following coherence conditions are satisfied:

$$\begin{array}{ccc} [A, A'] \otimes [B, B'] & \xrightarrow{\otimes_{ABA'B'}} & [A \otimes B, A' \otimes B'] \\ & \searrow^{\mathcal{L}_{[A,A'][B,B']}} & \downarrow \beta_{ABA'B'} \\ & & [A, A'] \boxtimes [B, B'] \end{array}$$

We will continue to use the string diagrammatic calculus for symmetric monoidal categories to absorb the manipulable $\otimes$ monoidal product, relegating terms involving $\boxtimes$ the cp-tensor product to be expressed symbolically. Given this convention the coherence conditions given above for complete-parallelism may be expressed diagrammatically:



We now example the morphisms of super-monoidal categories which furthermore preserve cp-structure.

Definition 7 (Morphism of cp-Super Monoidal Categories)**.** *A morphism of cp-Super Monoidal Categories is a morphism $(\mathcal{F}^C, \mathcal{F}^V, \mathcal{F}_{AB})$ of super-monoidal categories such that the functor $\mathcal{F}^V$ is eqipped to a monoidal functor*

$$\mathcal{F}_{\boxtimes} := (\mathcal{F}^V, \phi_{\boxtimes}, \epsilon_{\boxtimes}) : (\mathcal{V}, \boxtimes_V, I_V) \longrightarrow (\mathcal{V}, \boxtimes_V, I_V)$$

and such that the following functors commute

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\mathcal{F}^H_{\boxtimes}} & \mathcal{V}' \\ {\scriptstyle \mathcal{L}_V}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{L}_{V'}} \\ \mathcal{V} & \xrightarrow{\mathcal{F}^H_{\otimes}} & \mathcal{V}' \end{array}$$

and such that the following coherence conditions

$$\begin{array}{ccccc} & & \mathcal{F}[A,A']\mathcal{F}[B,B'] & & \\ & \swarrow^{\mathcal{F}_{A,A'} \boxtimes \mathcal{F}_{B,B'}} & & \searrow^{\phi^{\boxtimes}_{[A,A'][B,B']}} & \\ [\mathcal{F}A, \mathcal{F}A'][\mathcal{F}B, \mathcal{F}B'] & & & & \mathcal{F}[A,A'][B,B'] \\ \big\downarrow{\scriptstyle \beta_{\mathcal{F}A,\mathcal{F}A',\mathcal{F}A',\mathcal{F}B'}} & & & & \big\downarrow{\scriptstyle \mathcal{F}\beta_{A,B,A',B'}} \\ [\mathcal{F}A\mathcal{F}B, \mathcal{F}A'\mathcal{F}B'] & & \circlearrowright & & \mathcal{F}[AB, A'B'] \\ \big\downarrow{\scriptstyle i \boxtimes \phi^{\otimes}_{A',B'}} & & & & \big\downarrow{\scriptstyle \mathcal{F}_{AB,A'B'}} \\ [\mathcal{F}A\mathcal{F}B, \mathcal{F}A'B'] & & \xrightarrow[\phi^{\otimes}_{A,B} \boxtimes i]{} & & [\mathcal{F}AB, \mathcal{F}A'B'] \end{array}$$

are satisfied: for which all tensor products such as $A \boxtimes_{\mathcal{V}} B$ and $A \otimes_{\mathcal{C}} B$ along with object subscripts have been suppressed whenever the interpretation is clear.

Theorem 6 (Category of Completely Parallel Theories)**.** *The cp-super monoidal categories and the morphisms between them define a category*

Proof. Given in the Appendix □

Example 3 (1st Order Types in Higher Order Causal Categories)**.** *Taking $\mathcal{D}$ to be some pre-causal category, then the categories* **Caus**$[\mathcal{D}]$ *and* **Caus**$_1[\mathcal{D}]$ *define a* **Caus**$[\mathcal{D}]$ *cp-Super Monoidal Category* **Caus**$_1[\mathcal{D}]$*. For every A, A', B, B' in* **Caus**$_1[\mathcal{D}]$ *then taking the $\boxtimes$ of* **Caus**$[\mathcal{D}]$ *to be the tensor product the morphism*

$$\kappa_{AA'BB'} : [A, A'] \boxtimes_V [B, B'] \cong [A \otimes_C B, A' \otimes_C B']$$

is given by the following sequence of natural isomorphisms noted in [54].

$$[A, A'] \boxtimes_V [B, B'] \tag{19}$$
$$\cong [A, I] \boxtimes A' \boxtimes [B, I] \boxtimes B' \tag{20}$$
$$\cong [A, I] \boxtimes [B, I] \boxtimes A' \boxtimes B' \tag{21}$$
$$\cong [[[A, I] \boxtimes [B, I], I], A' \boxtimes B'] \tag{22}$$
$$\cong [A \otimes B, A \otimes B] \tag{23}$$

Crucially as the above is a sequence of natural isomorphisms, their composition κ is natural in all four components, this is furthermore easily witnessed by the fact that in the underlying category $\mathcal{D}$ κ is in fact the identity morphism. In $\mathbf{Caus}[\mathcal{D}]$ *one can show that the identity of the underlying category $\mathcal{D}$ is a morphism*

$$\mathbf{id} : X \otimes Y \to X \boxtimes Y$$

and so can be immediately checked to satisfy all of the required coherence conditions for a lax monoidal functor, along with the coherence required with κ and the parallel composition process $\otimes$.

5 Unification of Lower and Higher Order Physics: Linked Super Monoidal r-Categories

In this section we demonstrate that CSMC's can be fully characterised as a simple sub-class of theories of super-maps. Every Closed SMC is equivalent to a $\mathcal{V}$-Super Monoidal Category $\mathcal{C}$ satisfying two basic operational conditions that ought to be satisfied in any unified theory of lower and higher order physics.

- All processes in the theory $\mathcal{C}$, whether they be higher or lower order, can be manipulated inside $\mathcal{C}$
- Objects exist only to encode a space of potential states - there should be an identification between a system A and the higher order system $[I, A]$ representing the states of A.

Conceptually: the first condition models the assumption that it is the same agents that can perform processes, super-processes, and so on.

Definition 8. *A* Self-Super-Monoidal Category *is a $\mathcal{C}$-Super Monoidal Category $\mathcal{C}$.*

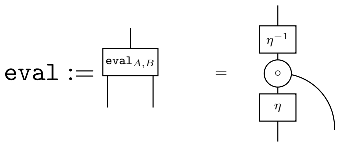
The second condition is captured by an additional notion of being *linked*, the possibility of moving between lower and higher order, moving between the part of the theory which concerns processes and the part of the theory which contains the super processes and even higher into the part of the theory which contains super-super-processes and so on. The notion of a self-Super-Monoidal category does not guarantee such an interaction between its sub-theories, indeed it only captures the notion that lower and higher order processes can peacefully co-exist next to eachother (inside a single symmetric monoidal category). The most basic way to introduce a linking interaction between lower and higher order processes is to require that access to a resource of type A in some theory, could just as-well be seen as access to a resource of type $[I, A]$, which after-all, should represent nothing other than the space of states of type A.

Definition 9. *A* Linked Super-Monoidal Category *is a Self-Super-Monoidal Category $\mathcal{C}$ equipped with a monoidal natural isomorphism $\eta : \mathcal{I}(-) \longrightarrow [I, -]$.*

Conveniently it turns out that Linked Super-Monoidal Categories are equivalent to a familiar and well-studied notion, that of a closed symmetric monoidal category.

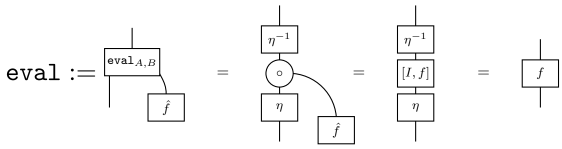
Theorem 7 (Characterisation of Closed Monoidal Categories)**.** *There exists a closed monoidal structure on a category $\mathcal{C}$ if and only if there exists a linked $\mathcal{C}$ Super-Monoidal r-Category $\mathcal{C}$*

Proof. We begin by showing that the first statement implies the second. Let $\mathcal{C}$ be a complete Super-Monoidal Category. Then, to each pair $A, B \in \mathcal{C}$ assign the candidate for evaluation



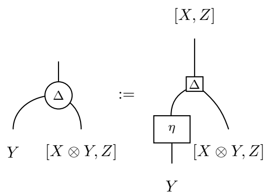
(24)

Since every $\circ$ is completely injective by assumption, so is every `eval`. Since $\eta : A \to [I, A]$ is a natural isomorphism for any $f \in \mathcal{C}(A, B)$ there exists a morphism $\hat{f}$ such that



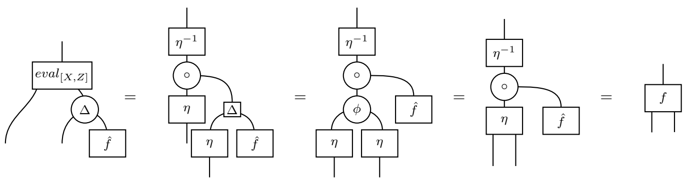
(25)

One can apply the isomorphism η to the partial insertion operation to generate a partial insertion using a lower level type Y as opposed to the higher level type $[I, Y]$.



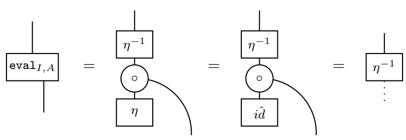
(26)

This partial insertion operation can be used to construct the curried version of any process f from its static version $\hat{f}$, since



(27)

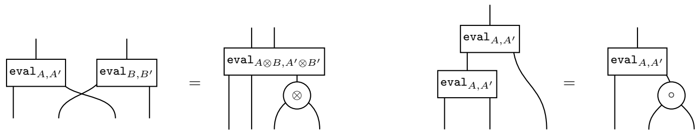
It follows that for every process f its curried version exists, that is, the co-universal arrow definition of a closed symmetric monoidal category is satisfied. All that remains is to show that `eval` is an isomorphism, note that since $\eta : (Id, id, id) \to ([I, -], \phi_1, \hat{id})$ is monoidal, it follows that $\eta = \eta \circ id_I = \hat{id}$. From this we see that,



(28)

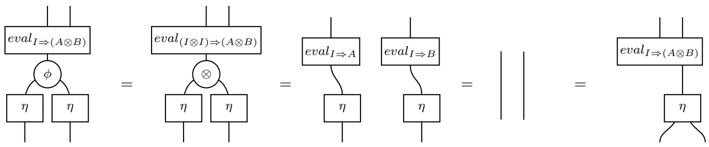
and so `eval` must indeed be an isomorphism.

Now we demonstrate the converse, namely that a closed SMC such that every $\texttt{eval}_{[I,A]}$ is an isomorphism is a complete Super-Monoidal Category. Let $\mathcal{C}$ be a closedSMC, then there exist sequential and parallel composition morphisms defined as adjuncts to circuits of evaluation morphisms. Concretely the definition enforces that there must exist processes $\otimes$ and $\circ$ satisfying,



(29)

which for satisfy the coherence conditions for a SMSC. Finally one has to show that the category is linked, that is - that there is a monoidal natural isomorphism $A \cong [I, A]$ for the induced functor $[I, -]$, indeed up to unitor the inverse η of $\texttt{eval}_{I\Rightarrow A}$, being an isomorphism by assumption, is such a candidate. $\texttt{eval}_{I\Rightarrow A}$ is natural for any closed monoidal category, so η being its inverse is immediately also natural. Furthermore η is easily checked to be monoidal.



(30)

This completes the proof. □

It follows that any example of a closed monoidal category, of which there are many in the literature could so far present a candidate higher-order theory of physics. As such this result represents a baseline, a bare minimum requirement for a higher order theory, leaving open the challenge of importing additional physical principles which might distinguish higher order physical theories from their non-physical closed monoidal cousins just as the category of quantum processes can in fact be *entirely* operationally distinguished from all other symmetric monoidal categories [36]. The most familiar notion of a physical higher order theory in the literature is that of a theory which is constructed whilst imposing the preservation of some notion of causality, in which processes are required to either have unit trace, or preserve iterated notions of trace. The motivating example of such a theory is a higher order causal category [54], for which we present here a minor generalisation, in which the notion of compactness of a raw-material category $\mathcal{C}$ is relaxed, with the aim of being applicable to standard mathematical formalisation of infinite dimensional systems for which the bounded linear maps appear as states in the closed monoidal category of Banach spaces.

Definition 10 (Dual Set)**.** *For a set of states $c \subseteq \mathcal{C}(I, A)$ of an arbitrary category $\mathcal{C}$ the dual set c^* is defined by*

$$\pi \in c^* \iff \forall \rho \in c \quad \pi \circ \rho = 1$$

Similarly for a set of effects $c \in \mathcal{C}(A, I)$ the dual set c^* is defined by

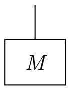
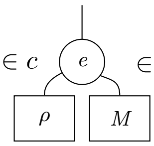
$$\rho \in c^* \iff \forall \pi \in c \quad \pi \circ \rho = 1$$

The notion of dual set is all that is required to construct a closed monoidal category which is deterministic (the only scalar being 1) from any raw-material closed monoidal category. We will in general write the evaluation morphism of a closed monoidal category by the symbol e.

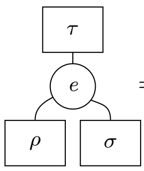
Example 4 (Higher Order Deterministic Categories)**.** *For every full and causally pointed subcategory $\mathcal{X}$ of a closed monoidal category $\mathcal{C}$ there exists a closed monoidal category $\mathbf{Det}[\mathcal{C}]$ in which objects are given by pairs (A, c) in which A is an object of $\mathcal{C}$ and c is a set of states $c \subseteq \mathcal{C}(I, A)$ such that $c^{**} = c$. The morphisms from $(A, c) \to (A', c')$ are taken to be those morphisms $f : A \to A'$ such that for each $\rho \in c$ then $f \circ \rho \in c'$. This category is symmetric monoidal, with tensor product given by $(A, c) \boxtimes (A', c') := (A \otimes A', (c \otimes c')^{**})$ where*

$$c \otimes c' := \{\rho \otimes \rho' : \rho \in c \texttt{ and } \rho' \in c'\}$$

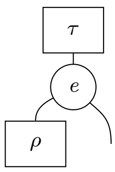
and the tensor product has its action on morphisms inherited from $\mathcal{C}$ I.E $f \boxtimes g := f \otimes g$. The tensor unit is given by $(I, \{1\})$. The proof that the above defines a symmetric monoidal category takes the identical steps to those of [54]. The category $\mathbf{Det}[\mathcal{C}]$ is furthermore closed monoidal, for every $(A, c), (A', c') \in \mathbf{Det}[\mathcal{C}]$ there exists the object $(A \Rightarrow A', c \Rightarrow c')$ where

 $\in (c \Rightarrow c') \iff \forall\!: \rho \in c$  $\in c'$ (31)

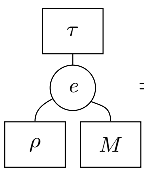
For every $c = c^{**}$ and $c' = c'^{**}$ the set $c \Rightarrow c'$ indeed satisfies $(c \Rightarrow c')^{**} = c \Rightarrow c'$ since first of all for any set $s \subseteq s^{**}$ but furthermore for any $M \in (c \Rightarrow c')^{**}$ then for every $\pi \in (c \Rightarrow c')^{*}$ it follows that $\pi \circ M = 1$, in turn this implies that since for every $\tau \in c'^{*}$ and $\rho \in c$ then for every $\sigma \in (c \Rightarrow c')$

 $= \quad 1$ (32)

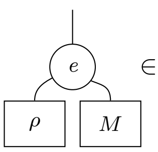
which entails that

 $\in \quad (c \Rightarrow c')^{*}$ (33)

which entails that for every $M \in (c \Rightarrow c')^{**}$ then

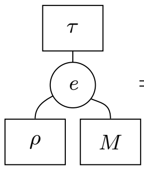
 $= \quad 1$ (34)

which entails that

 $\in \quad c'^{**} = c'$ (35)

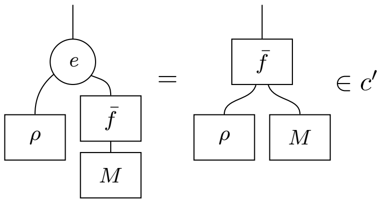
where the last equality follows by closure of c'. The evaluation morphisms of the closed monoidal structure of $\mathcal{C}$ are inherited into a closed monoidal structure for $\mathbf{Det}[\mathcal{C}]$, firstly the evaluations are indeed morphisms of the correct type $e : (A, c) \boxtimes (A \Rightarrow A', c \Rightarrow c') \to (A', c')$ since for every $\rho \in c$

every $M \in (c \Rightarrow c')$ *and every* $\tau \in c'^*$



$$= 1 \tag{36}$$

which implies that $\tau \circ e \in (c \otimes (c \Rightarrow c'))^*$*. In turn this implies that for every* $\tau \in c'^*$ *and every* $K \in (c \otimes (c \Rightarrow c'))^{**}$ *then* $\tau \circ e \circ K = 1$ *and so* $e \circ K \in c'^{**} = c'$*. Finally for every* $f : (A, c) \boxtimes (Z, z) \to (A, c')$ *its currying* $\hat{f} : Z \to (A \Rightarrow A')$ *exists as a morphism* $f : (Z, z) \to (A \Rightarrow A', c \Rightarrow c')$ *since for every* $\sigma \in z$ *and every* $\rho \in c$ *then*



(37)

which is precisely the condition that $\bar{f} \circ \sigma$ *be a member of* $(c \Rightarrow c')$*. Uniqueness is inherited from uniqueness in* $\mathcal{C}$*. This is sufficient to entail that* $\mathbf{Det}[\mathcal{C}]$ *be closed monoidal.*

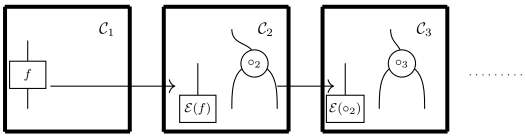
Curiously we note that there is no reason to expect the result of the generalised $\mathbf{Caus}[\mathcal{C}]$ construction to be $*$-Autonomous, however the property of $*$-Autonomy is lifted from $\mathcal{C}$ to $\mathbf{Caus}[\mathcal{C}]$ whenever it is present in $\mathcal{C}$.

Example 5 (Day Convolution)**.** *For every* $\mathcal{V}$*-Enriched Category* $\mathcal{C}$ *the enriched presheaf category* $[\mathcal{C}^{op}, \mathcal{V}]$ *is a closed monoidal category with respect to the day convolution product* $\otimes_{Day}$*. There is a full faithful strong monoidal functor* $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{V}]$ *meaning that every physical theory can be viewed as a full subcategory (in the invariant sense) of a higher order theory. In future work it will be of interest to compare the kinds of higher order functions that exist within the day convolution of the category of causal quantum processes and compare them with those which exist in the category* $\mathbf{Caus}[\mathbf{CPM}[\mathbf{fHilb}]]$*. Furthermore this leaves open the question of the kinds of theories which are in some sense initial or terminal, and the possibility to contribute to the discourse over the supermaps which should be regarded as "reasonable" by considering those higher order theories which satisfy some universal properties analogous to those consdiered for standard quantum theory in [Staton].*

The notion of a higher order category which is deterministic but not causal in the usual sense may be useful for modelling higher order processes in time-symmetric settings [74]. Having pinned down the higher-order part of higher order physical theories, it will be of interest in future work to analyse the interaction of closed monoidal structure with other operational notions, in particular the introduction of convex mixtures and generalised probabilistic structures [58].

6 Towers of Super-Physical Theories

In the remainder of this manuscript we present a no-go theorem, on the kind of theories that can result from the gluing together of a tower of physical theories, demonstrating that if a theory consists of essentially no more than a tower of sufficiently well behaved physical theories *and* it is possible within the theory, to move between the layers of the tower *then* is **must** be the case that such a theory is closed monoidal. The formal object of study about which the above theorem is proven is similar to the notion of a co-limit but the authors suspect it to be more general. Following the development of the field of higher order physics, after the introduction of supermaps, it is natural to develop theories in which those supermaps themselves may be subject to manipulation by even *higher* order theories. This research direction has culminated in the construction of complete higher order physical theories for finite dimensional GPTs [7, 79] and pre-causal categories [54]. We now present the notion of a family of theories over a base theory, each a theory of supermaps over the theory that precedes it. The base theory represents a given physical theory, such as quantum or classical probability theory. The second layer represents a theory of supermaps, the third layer a theory of super-supermaps, an so on, as illustrated in the following picture



(38)

The ultimate goal of introducing this construction is for the specification of a complete higher order theory, into which such a sequence will embed. Mathematically, a hierarchy of higher order physical processes is represented by an *ascending sequence of Super-Monoidal Categories*

Definition 11. *An* ascending sequence of Super-Monoidal r-Categories $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N)$ *is a sequence of Symmetric Monoidal Categories such that for every i there exists a $\mathcal{C}_{i+1}$-Super Monoidal r-Category $\mathcal{C}_i$.*

In such a sequence, the category $\mathcal{C}_i$ is "encoded" into the higher level $\mathcal{C}_{i+1}$ by the monoidal *raising* functor $\mathcal{R}_i^{i+1}(-) := [I_i, -]$. It will be convenient to define the following compact notation for the induced encoding (full faithul braided monoidal functor) from level i to level $j > 1$.

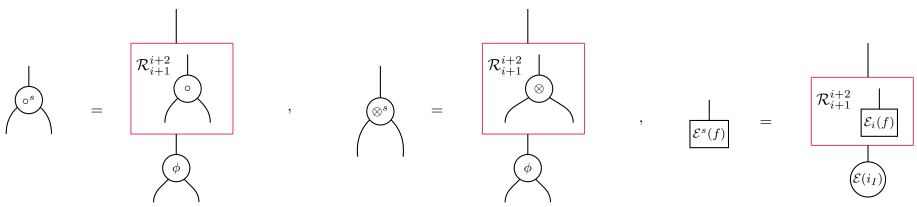
$$\mathcal{R}_i^j : \mathcal{C}_i \longrightarrow \mathcal{C}_j \qquad \mathcal{R}_i^j := \mathcal{R}_{j-1}^j \circ \mathcal{R}_i^{j-1}$$

Theorem 8. *Let $\mathcal{C}_i$ be an* ascending sequence of Super-Monoidal r-Categories *then for every $j > i$ there exists a $\mathcal{C}_j$-Super Monoidal r-Category $\mathcal{C}_i$*

Proof. There are in fact at least two natural ways to construct a $\mathcal{C}_{i+2}$-Super Monoidal r-Category $\mathcal{C}_i$. Firstly for each $A, B \in \mathcal{C}_i$ define

$$[A \rightsquigarrow^s B] := \mathcal{R}_{i+1}^{i+2}([A, B]) \in \mathcal{C}_{i+2} \tag{39}$$

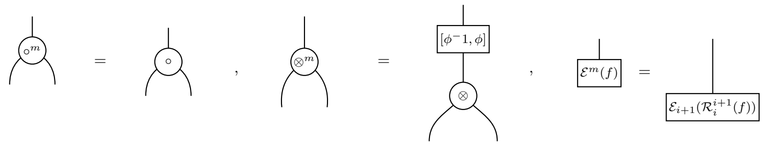
and define



the lifting of the composition morphisms of $\mathcal{C}_{i+1}$. Then the satisfaction of all defining conditions of a super-monoidal category are given by the lifting of those Conditions by functorality. Secondly, for each $A, B \in \mathcal{C}_i$ define

$$[A \rightsquigarrow^m B] := [\mathcal{R}_i^{i+1}(A), \mathcal{R}_i^{i+1}(B)] \in \mathcal{C}_{i+2} \tag{40}$$

and define the composition morphisms $\bigcirc^m, \bigotimes^m$ and the bijection $\mathcal{E}^m$ by



The proof that this defines a super-monoidal category is given in the appendix. By iterating either of the above constructions a $\mathcal{C}_j$-Super Monoidal r-Category $\mathcal{C}_i$ may be constructed for any $j > i$. □

As previously noted, there is a sense in which the agents inhabiting layer $\mathcal{C}_j$ are strictly more powerful than the inhabitants of $\mathcal{C}_{i<j}$, in the sense that each $\mathcal{C}_{i<j}$ may be embedded into $\mathcal{C}_j$. For any finite sequence $\mathcal{C}_i$ of length n, the final category $\mathcal{C}_n$ may be seen as the arena in which agents my perform manipulations over any category in the preceding sequence. This notion does not extend so simply to infinite sequences in which there is no member of the sequence that can be interpreted as the theory which contains all of the others, to cope with this issue, in which one would like to define a category which contains the infinitely powerful agents that may implement processes from any member of an infinite sequence we will introduce the notion of a `Merger` for a sequence, which will trivialise for all but infinite sequences. Before constructing from such a hierarchy of super-theories the definition of a Merger, a property of consistency along such a sequence should be addressed. There exists a family of morphisms

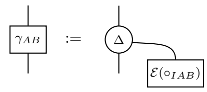
$$\gamma_{A,B} : [A \rightsquigarrow^s B] \rightarrow [A \rightsquigarrow^m B] \tag{41}$$

which can be used to construct a super monoidal r-functor.

Theorem 9. *Let $(\mathcal{C}_{(i)})$ be an ascending sequence of Super-Monoidal categories such that every $[I, -]$ is full. Then, for all i, there exists a super-monoidal r-functor*

$$\gamma^i : (\mathcal{C}_{i+2}, \mathcal{C}_i, \mathcal{E}^s, [- \rightsquigarrow^s -]) \longrightarrow (\mathcal{C}_{i+2}, \mathcal{C}_i, \mathcal{E}^m, [- \rightsquigarrow^m -]) \tag{42}$$

defined by the functors $\gamma^C(-) = \mathcal{I}_{\mathcal{C}_i}(-)$ and $\gamma^V(-) = \mathcal{I}_{\mathcal{C}_{i+2}}(-)$ along with the family



(43)

Proof. Given in Appendix. □

For the next step in our construction we will require a stronger condition, namely that γ be an isomorphism, meaning that the above enrichments of $\mathcal{C}_i$ are equivalent.

Definition 12. *An ascending sequence of nested Super-Monoidal Categories is $(\mathcal{C}_i)_{i \in I}$* fully coherent *if*

- *for all $i \in I$ $\mathcal{C}_{i+1}$ is a Super-Monoidal category for $\mathcal{C}_i$, and*
- *for all $i \geq 3$ the canonical super monoidal r-functor γ^i is full and faithful.*

6.1 Theories Consisting of Coherent Towers are Closed Monoidal

The mathematical notions are now in place for the definition of a Merge, consisting in all the data that should be supplied in the definition of a theory of all-powerful higher order transformations which act upon lower order objects, higher order objects, and mixes between them. Given a sequence of higher order theories, the data which will be used to specify a physical theory, all that remains is the notion of a theory $\mathcal{C}$ into which such a sequence $\mathcal{C}_{(i)}$ is embedded. Such an embedding for a generic sequence $\mathcal{C}_{(i)}$ is captured categorically by a sequence of full and faithful functors $\mathcal{F}_i : \mathcal{C}^{(i)} \to \mathcal{C}$. For $\mathcal{C}$ to represent a theory which contains $\mathcal{C}^{(i)}$ and *nothing more* a further condition of essential surjectivity should be imposed on the union (co-product) functor

$$\coprod_i \mathcal{F}_i : \mathcal{C}^{(i)} \to \mathcal{C}$$

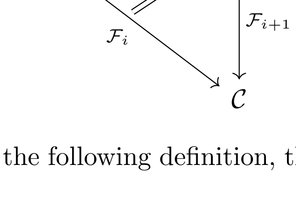
Finally we impose the condition that there be a way to freely move between layers of the theory. The most basic notion of a linking between levels is via an isomorphism $A \cong [I, A]$. There should be no practical distinction between having access to A and having access to the space of states $[I, A]$ on A. Furthermore there ought not be a practical difference between the performing of $f : A \to B$ or instead the conversion of A to $[I, A]$ and the consequent application of $[I, f]$ ("f" interpreted as a map which applies f to the states of A to produce states of B). Formally the equivalence between these actions, when consistent with the monoidal structure, is captured by the existence of a monoidal natural isomorphism

$$\eta^i_{i-1} : \mathcal{F}_{i-1}(-) \longrightarrow \mathcal{F}_i \circ \mathcal{R}^i_{i-1}(-)\,, \tag{44}$$

In short, η^i_{i-1} provides a witness for the equivalence between A and $[I, A]$ inside $\mathcal{C}$. For ease of notation we will denote the inverse $(\eta^i_{i-1})^{-1}$ by η^{i-1}_i when needed. The existence of a natural isomorphism η^i_{i-1} for each i is can be concisely phrased in the language of 2-Categories, it is precisely the requirement that $\mathcal{C}$ be a 2-Cone in the 2-Category **ffSymCat** of

- Symmetric monoidal categories
- Full Faithful Strong Monoidal Functors
- Monoidal Natural Transfomations

That is, for a diagram $\mathcal{D}$ in **ffSymCat** given by a coherent sequence of super-monoidal categories, and the monoidal functors $\mathcal{R}^{i+1}_i : \mathcal{C}_i \longrightarrow \mathcal{C}_{i+1}$ between them, a cone over $\mathcal{D}$ is an "apex" category $\mathcal{C}$ equipped with a family of functors $\mathcal{F}_i : \mathcal{C}_i \to \mathcal{C}$ such that each of the following triangles commutes up to a monoidal natural isomorphism η^{i+1}_i.



The above discussion culminates in the following definition, that of a complete higher order physical theory - termed a `Merger`.

Definition 13. *A Merger for a coherent sequence of Super-Monoidal categories $\mathcal{C}^{(i)}$ is a 2-Cone $(\mathcal{F}_i : \mathcal{C}_i \longrightarrow \mathcal{C})$ over the diagram*

$$\ldots \qquad \mathcal{C}_{i-1} \xrightarrow{\mathcal{R}^{i}_{i-1}} \mathcal{C}_i \xrightarrow{\mathcal{R}^{i+1}_{i}} \mathcal{C}_{i+1} \qquad \ldots$$

in **ffSymCat** *such that*

- *The coproduct functor $\coprod_i \mathcal{F}_i$ is essentially surjective*

A Merger is furthermore termed "n-th order" if the sequence has length n.

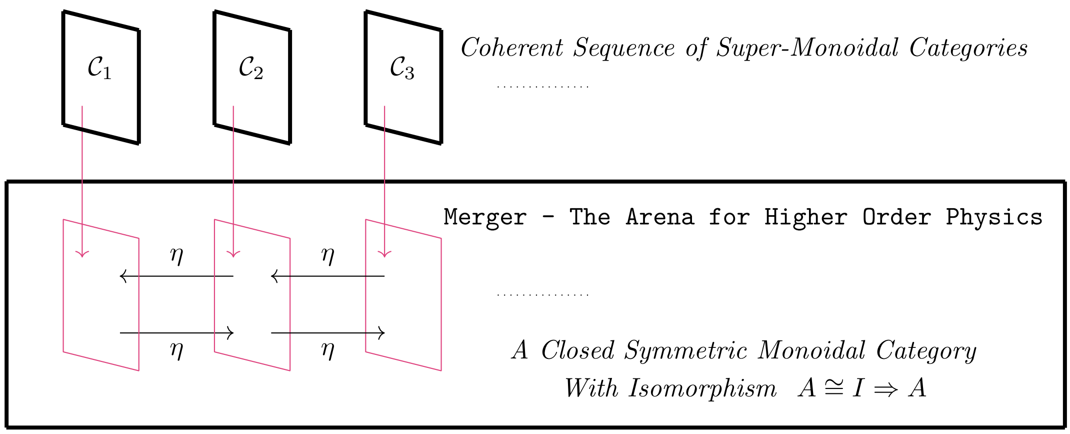
For any sequence of finite order the notion of a Merger is essentially trivial, given a sequence of order n one can simply construct a cone of the above type by taking $\mathcal{C} = \mathcal{C}_n$ and taking $\mathcal{F}_i := \mathcal{R}^n_i$.

To a category-theorist it may seem a little unusual to see the notion of a theory which contains an infinite sequence of theories as anything other than a co-limit of that sequence. In the definition of a merger we trade mathematical familiarity for physical intuition, this is with the ultimate goal of keeping transparent the conceptual assumptions made about higher order theories which will eventually lead to the derivation of closed monoidal structure for an infinite order higher order theory. it may in principle be true that mergers are equivalent to co-limits, we leave that question for future work.

The primary technical contribution of this manuscript is the discovery that the apex of any ∞-Order Merger possesses a simple categorical property, it must be a closed monoidal category.

Theorem 10. *The apex $\mathcal{C}$ of any Merger of infinite order is a complete super-monoidal category, in other words $\mathcal{C}$ is a CSMC such that every $e_{I \Rightarrow A}$ is an isomorphism.*

This result is intuitively captured in the following schematic diagram, a complete theory of higher order physics is one which contains an infinite hierarchy of theories, *and* in which agents can move between the layers of the hierarchy.



This simple to state categorical property on the arena in which higher order physics must take place, provides an easy starting position, from which the consequences of basic physical principles in *higher order* physics can be explored. First steps in this direction of research are taken in [80], in which an interaction between the strength of spatial correlations, determinism, and the possibility of signalling between parties is observed.

The culmination of the previous two sections, is an operational justification for the treatment of closed monoidal categories, in particular with the above stated canonical isomorphisms, as a key structural feature of higher order physics.

7 Conclusion

Presented in this manuscript is a mathematical framework, based on the notion of a monoidally enriched category, developed with the aim to capture the notion of a higher order physical theory. This framework is analogous in attitude to the process theory framework for standard physics based on the notion of a symmetric monoidal category. The framework accounts separately for theories with independently manipulable tensor product spaces of processes and theories which are *completely parallel* in the sense that the processes of the theory may be applied to part of *any* bipartite process. The framework permits easy phrasing of the notion of an iterated or internal higher order theory of physics, the inevitability of closed monoidal structure for such higher order theories is illustrated by two results: That Linked higher order theories *characterise* closed monoidal categories, and that categories into which infinite towers of higher order theories are suitably embedded, are always closed monoidal.

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Appendix

A Preliminary Definitions

Definition 14. *A lax monoidal functor* $(\mathcal{F}, \phi, \nu)$ *is a functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ *between categories* $\mathcal{C}$ *and* $\mathcal{D}$*, a morphism* $\phi_{A,B} : \mathcal{F}(A) \otimes \mathcal{F}(B) \to \mathcal{F}(A \otimes B)$ *for every pair of objects* A, B *and a morphism* $\nu : I_{\mathcal{D}} \to \mathcal{F}(I_{\mathcal{C}})$*, satisfying coherence conditions [60].*

The notion of equivalence between two representations is that of a monoidal natural transformation.

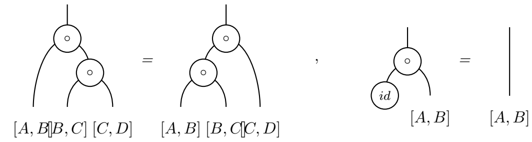
Definition 15. *A monoidal natural transformation* η *between monoidal functors* $(\mathcal{F}, \phi, \nu)$ *and* $(\mathcal{G}, \psi, \mu)$*, is a natural transformation satisfying coherence conditions [60].*

A key notion for the description of supermaps will be that of an enriched category:

Definition 16. *An* enriched category $(\mathcal{O}, \mathcal{C})$ *is a collection of objects* $\mathcal{O}$ *and a monoidal category* $\mathcal{C}$ *such that*

- *For every pair of objects in* $\mathcal{O}$ *there is an object* $[A, B]$ *in* $\mathcal{C}$

- *For every triple of objects A, B, B in $\mathcal{O}$ there is a composition morphism* $\circ : [A,B] \otimes [B,C] \to [A,C]$ *(with a minor abuse of notation in the use of the symbol $\circ$)*
- *For every object A of $\mathcal{C}$ there is a* unit morphism $id : I \to [A,A]$
- *The following conditions hold*



(45)

Any enriched category defines an ordinary category, in which the morphisms are the states of the enriched category, and the composition function is induced by the composition morphism.

Definition 17. *The* underlying category *of an enriched category $(\mathcal{O}, \mathcal{C}_2)$ is the category $\mathcal{C}_1$ where*

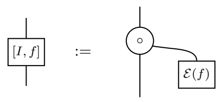
- *the objects in $\mathcal{C}_1$ are $Ob(\mathcal{C}_1) := \mathcal{O}$*
- *the hom-sets in $\mathcal{C}_1$ are $\mathcal{C}_1(A,B) := \mathcal{C}_2(I_2, [A,B])$, where I_2 is the tensor unit in $\mathcal{C}_2$*
- *the composition in $\mathcal{C}_1$ is given by $g \circ f := \circ(f \otimes g)$.*

The key notion for the description of a complete higher order physical theory will be that of a closed SMC:

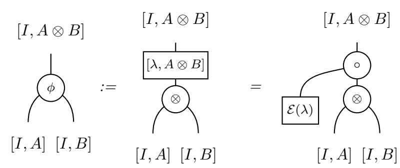
Definition 18. *An SMC $\mathcal{C}$ is* closed *if for every $A, B \in Ob(\mathcal{C})$ there exists an object $A \Rightarrow B$ and a morphism $\mathtt{eval}_{A\Rightarrow B} : A \otimes A \Rightarrow B \to B$, called the* evaluation morphism*, such that for all $f : A \otimes C \to B$ there exists a unique $\bar{f} : C \to (A \Rightarrow B)$ such that $\mathtt{eval}_{A\Rightarrow B} \circ (id \otimes \bar{f}) = f$.*

B Existence of a Monoidal Functor

We demonstrate that the structure of a $\mathcal{V}$-Symmetric Monoidal r-Category $\mathcal{C}$ is sufficient for the existence of a lax braided monoidal functor $[I, -] : \mathcal{C} \longrightarrow \mathcal{V}$. Where



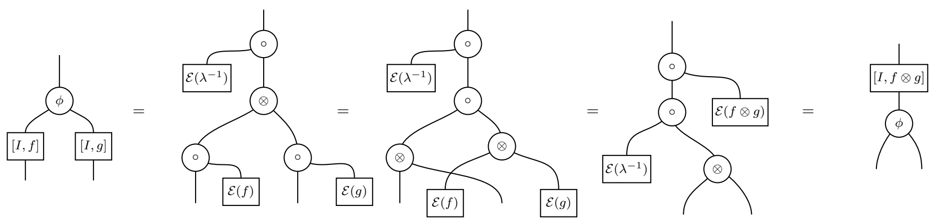
Theorem 11. *The data $([I,-], \phi_{AB}, \mathcal{E}(i_I))$ with the family $\phi_{AB} : [I,A] \otimes [I,B] \longrightarrow [I, A \otimes B]$ given by*



(46)

defines a lax monoidal functor.

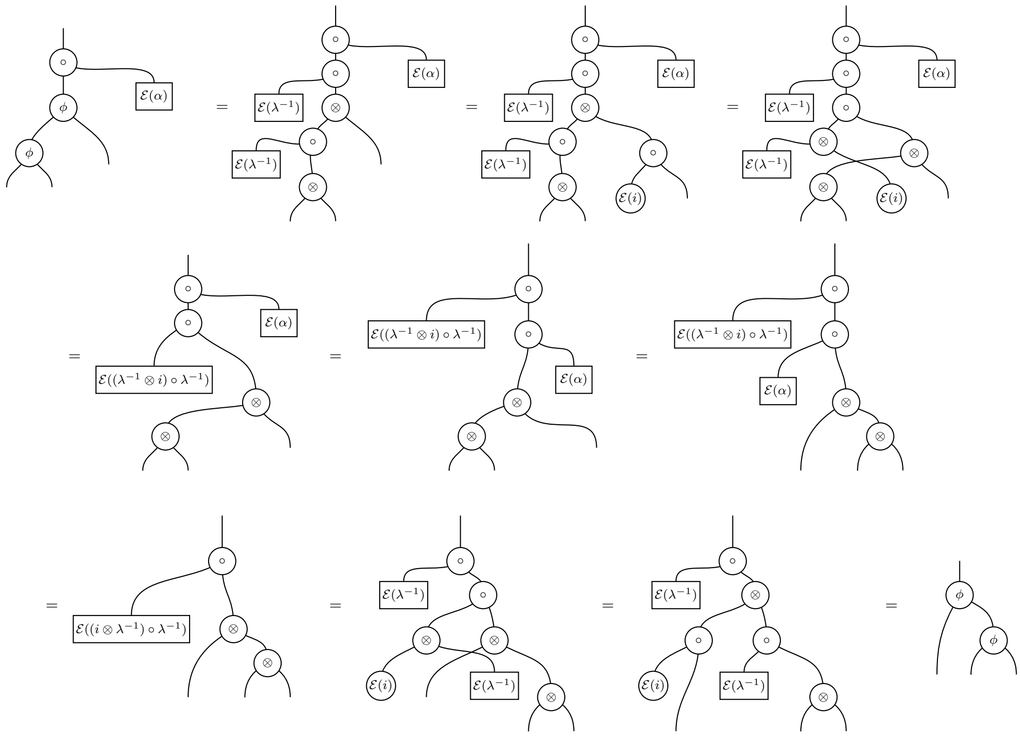
Proof. First of all it must be checked that the familly ϕ_{AB} form a natural transformation $\phi : [I,-] \otimes [I,-] \longrightarrow [I, - \otimes -]$.



The only coherence condition for a braided monoidal functor that does not follow trivially from the defining conditions of a $\mathcal{V}$-Symmetric Monoidal r-Category $\mathcal{C}$ is the coherence condition

$$\begin{array}{ccc}
([I,A]\otimes[I,B])\otimes[I,C] & \xrightarrow{\quad\alpha\quad} & [I,A]\otimes([I,B]\otimes[I,C]) \\
{\scriptstyle \phi\otimes i}\big\downarrow & & \big\downarrow{\scriptstyle i\otimes\phi} \\
[I,A\otimes B]\otimes[I,C] & & [I,A]\otimes[I,B\otimes C] \\
{\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\
[I,(A\otimes B)\otimes C] & \xrightarrow{\quad [I,\alpha]\quad} & [I,A\otimes(B\otimes C)]
\end{array}$$

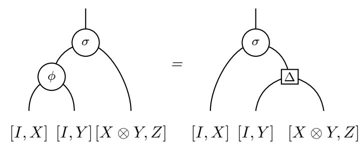
for the natural transformation ϕ. Indeed this condition follows by the following manipulations.



□

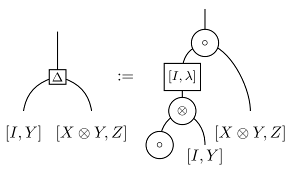
C The Existence of Partial Insertion

Lemma 3. *Let $\mathcal{C}_2$ be a Super-Monoidal category over $\mathcal{C}_1$. Then, there exists an operation Δ, named* partial insertion*, which satisfies the following condition:*



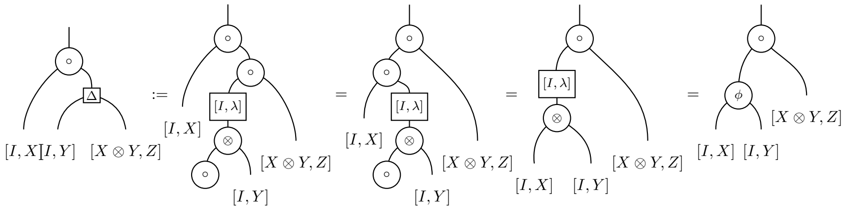
(47)

Proof. There indeed exists a canonical circuit of the correct type.



(48)

The lemma then follows from a combination of the coherence conditions of a Super-Monoidal category:



(49)

□

D The category of Super Monoidal r-Categories

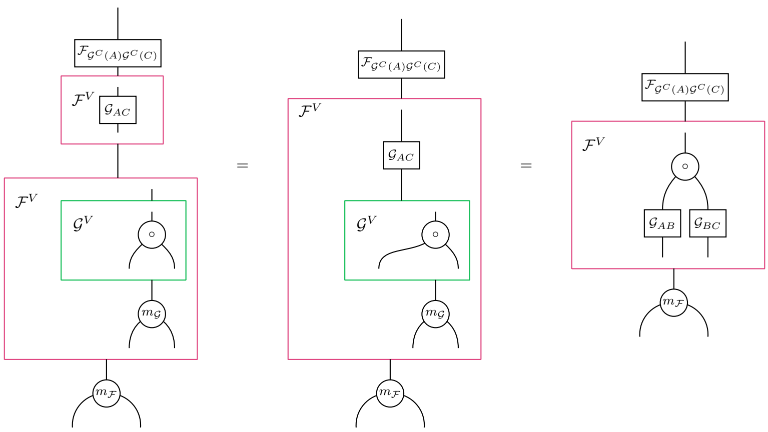
Here we demonstrate that the composition of Super Monoidal r-Functors is well defined.

Theorem 12. *Super Monoidal r-Categories and the Super Monoidal r-Functors between them define a category* **SupMonrCat**.

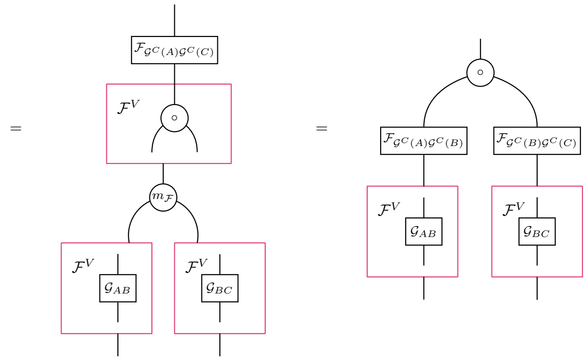
Proof. The composition $\mathcal{F} \circ \mathcal{G}$ of a super monoidal r-functor $\mathcal{F}$ and a super monoidal r-functor $\mathcal{G}$ is given by

- $(\mathcal{F} \circ \mathcal{G})^C := \mathcal{F}^C \circ \mathcal{G}^C$
- $(\mathcal{F} \circ \mathcal{G})^V := \mathcal{F}^V \circ \mathcal{G}^V$
- $(\mathcal{F} \circ \mathcal{G})_{AB} := \mathcal{F}_{\mathcal{G}^C(A)\mathcal{G}^C(B)} \circ \mathcal{F}^V(\mathcal{G}_{AB})$

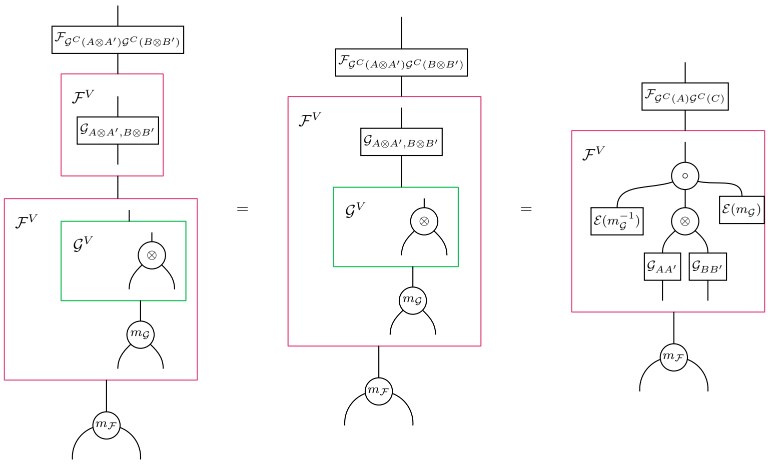
Now we demonstrate that the defining conditions of a super monoidal r-functor are satisfied. Firstly the preservation of sequential composition condition, first using functorality of $\mathcal{F}^V$ and super-monoidal functorality of $\mathcal{G}$,



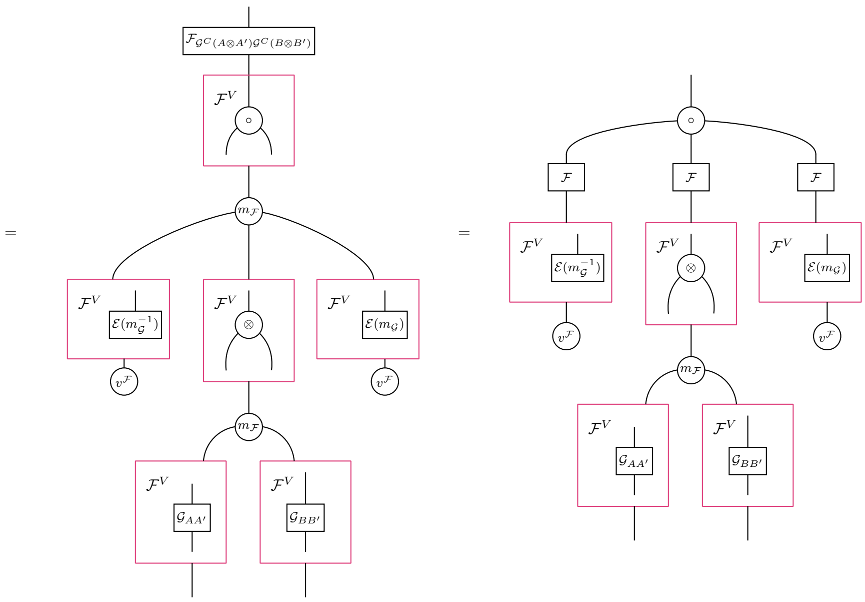
then using naturality of $m_{\mathcal{F}}$ and super-monoidal functorality of $\mathcal{F}$



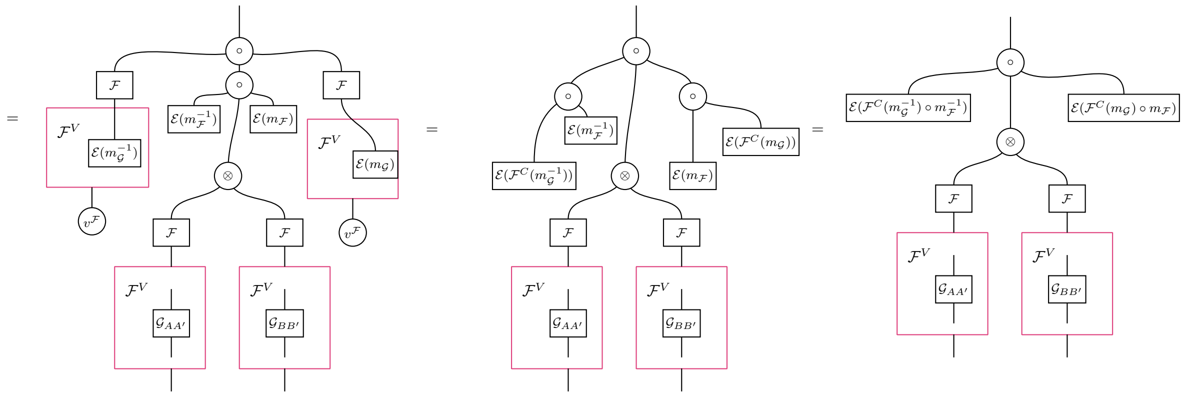
Next for the preservation of parallel composition condition



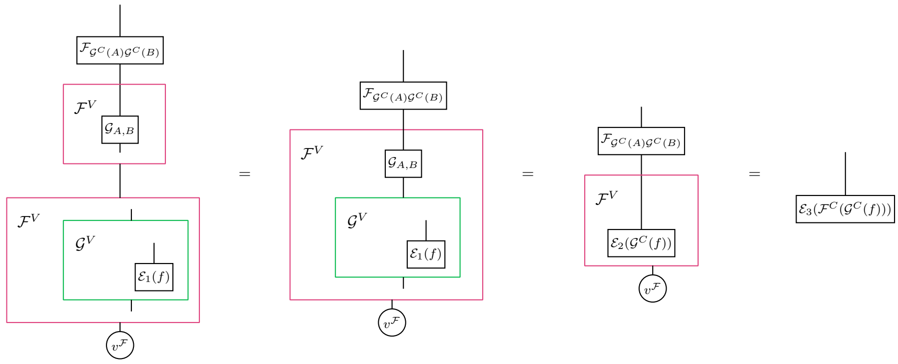
Using naturality of $m_{\mathcal{F}}$,



then using the composition of natural isomorphisms for monoidal functors.



All that remains is to demonstrate the state-based condition:



∎

E The Category of CP-Super Monoidal r-Categories

We show that cp-functors compose.

Theorem 13 (Category of Completely Parallel Theories)**.** *The cp-super monoidal categories and the morphisms between them define a category*

That the former condition for cp-functors is preserved by composition is guaranteed by the commutativity of the following diagram:

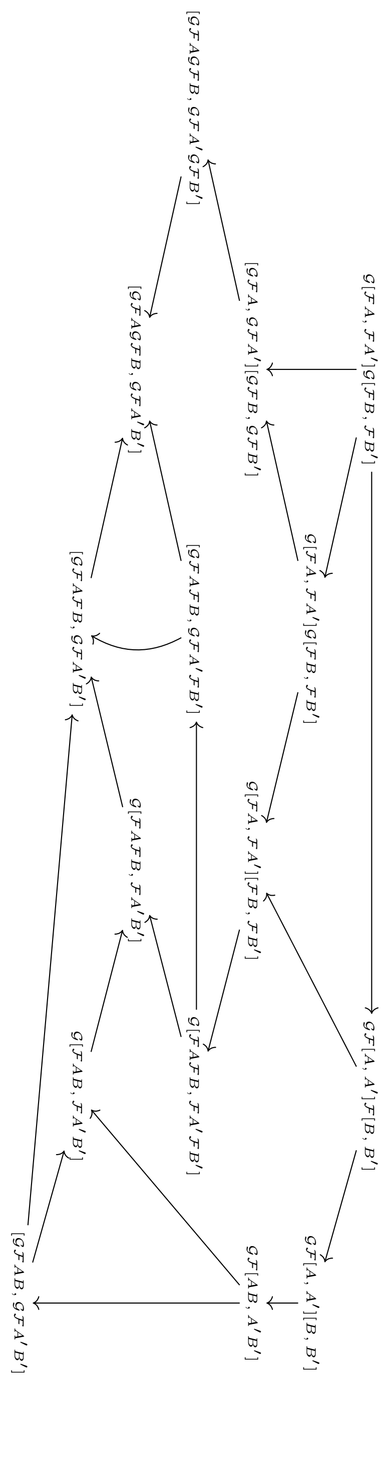
$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{\mathcal{F}^H_{\boxtimes}} & \mathcal{V}' & \xrightarrow{\mathcal{G}^H_{\boxtimes}} & \mathcal{V}'' \\ {\scriptstyle \mathcal{L}_V}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_{V'}} & & \downarrow{\scriptstyle \mathcal{L}_{V''}} \\ \mathcal{V} & \xrightarrow{\mathcal{F}^H_{\otimes}} & \mathcal{V}' & \xrightarrow{\mathcal{G}^H_{\otimes}} & \mathcal{V}'' \end{array}$$

the preservation under composition of the latter coherence requirement is more complicated. Using commutativity of the diagram:

we find that the following diagram commutes:

$$\begin{array}{ccc} [\mathcal{GFAFB}, \mathcal{GFA'FB'}] & \xleftarrow{\mathcal{G}_{\mathcal{FAFB},\mathcal{FA'FB'}}} & \mathcal{G}[\mathcal{FAFB}, \mathcal{FA'FB'}] \\ {\scriptstyle [i,\phi^{\otimes}_{\mathcal{FA'},\mathcal{FB'}}]}\downarrow & & \downarrow{\scriptstyle \mathcal{G}[i,\phi^{\otimes}_{A',B'}]} \\ [\mathcal{GFAFB}, \mathcal{GFA'B'}] & \xleftarrow[\mathcal{G}_{\mathcal{FAFB},\mathcal{FA'B'}}]{} & \mathcal{G}[\mathcal{FAFB}, \mathcal{FA'B'}] \end{array}$$

The above commutations along with the coherence of a *cp*-functor and the lifting of such coherence by $\mathcal{G}$ gives the following commutative diagram, which witnesses coherence of the composed functor $\mathcal{GF}$.

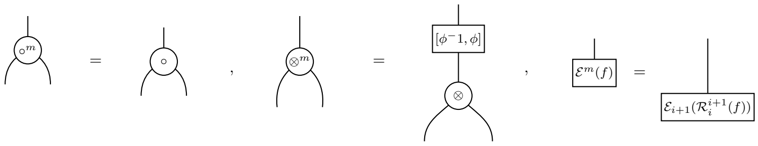


F Super Monoidal r-Categories Induced Along an Ascending Sequence of Super Monoidal r-Categories

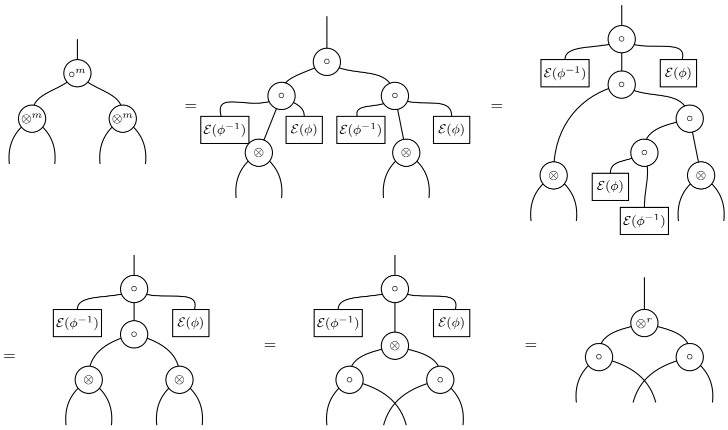
We demonstrate that for an ascending sequence $\mathcal{C}_i$ the assignment of for each $A, B \in \mathcal{C}_i$ an object,

$$[A \rightsquigarrow^m B] := [\mathcal{R}_i^{i+1}(A), \mathcal{R}_i^{i+1}(B)] \in \mathcal{C}_{i+2} \tag{50}$$

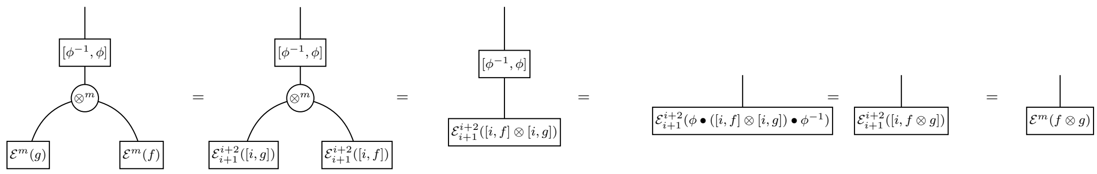
and the composition morphisms $\bigcirc^m, \bigotimes^m$ and the bijection $\mathcal{E}^m$ defined by



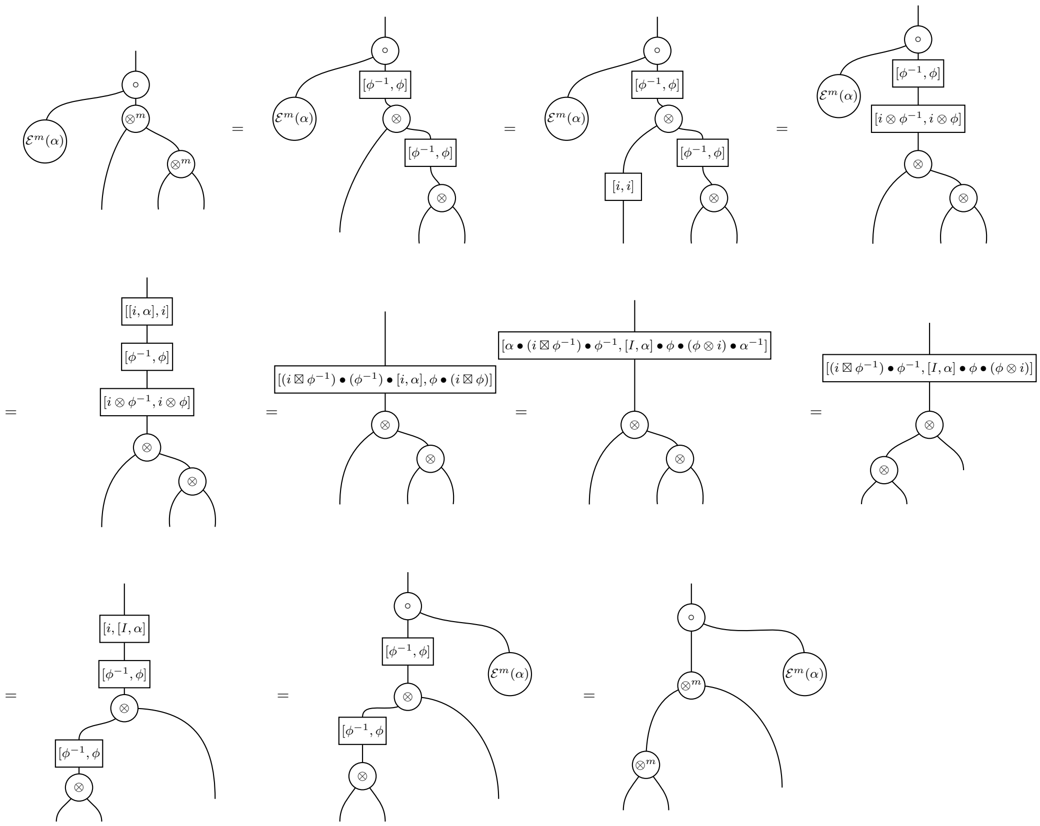
defines a super monoidal r-category. The $\mathcal{E}^m$ is a bijection follows immediately from the fact that by assumption $\mathcal{R}_i^{i+1}$ is full and faithful. What remains is to check all equational conditions:
`Bifunctorality`



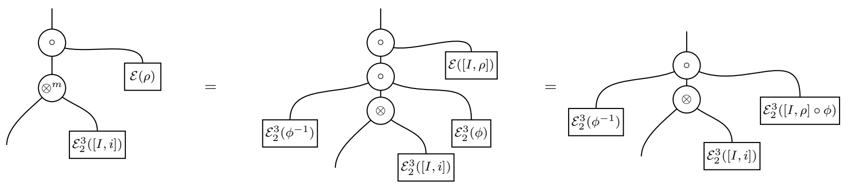
`Composition on States` The sequential composition condition is trivial, for parrallel composition



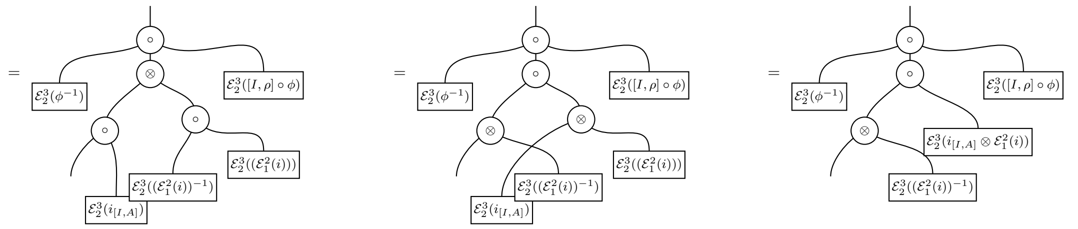
`Naturality` We check the associator condition.



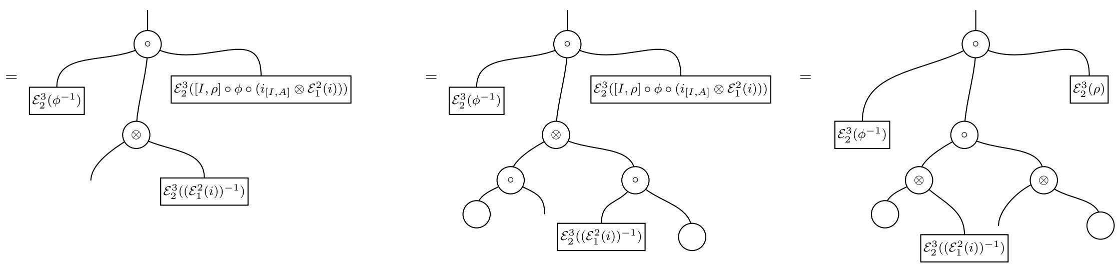
We confirm the unitor condition, first by exapanding out the relevent definitions,



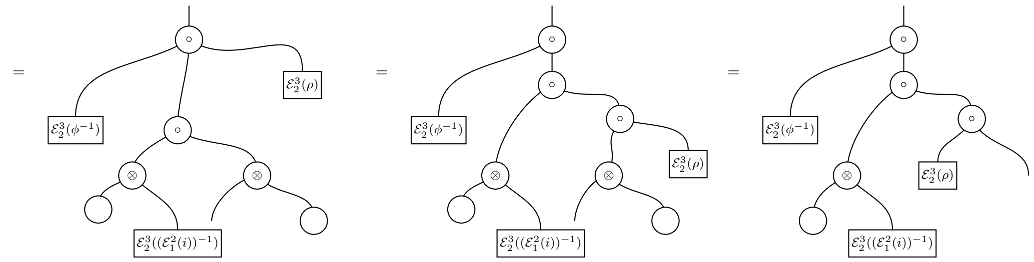
Then using bifunctorality,



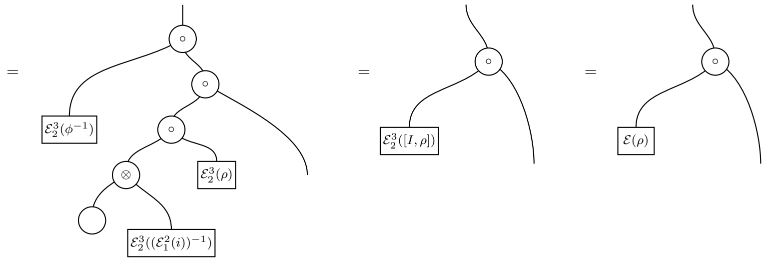
again using bifunctorality,



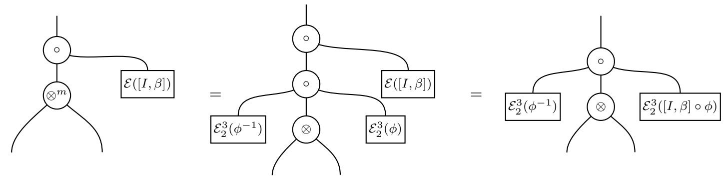
here using naturality of the $\mathcal{C}_{i+2}$-Super Monoidal category $\mathcal{C}_{i+1}$

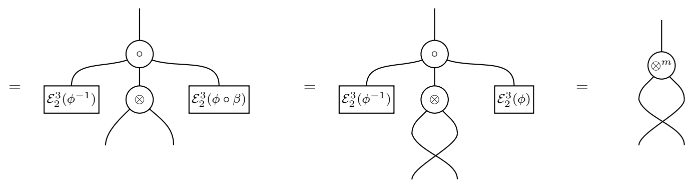


and then by further uses of associativity, the right hand side of the unitor condition is reached



the proof for the co-unitor is identical. `Braid` The braid rule follows by using the sequental composition rule to produce an internal version of the braid rule which then lifts to an external one.





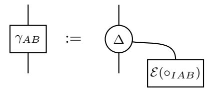
G Existence of a Super r-Functor

Here we prove the existence of the candidate enriched functor,

Theorem 14. *Let $(\mathcal{C}_{(i)})$ be an ascending sequence of Super-Monoidal categories such that every $[I,-]$ is full. Then, for all i, there exists a super-monoidal r-functor*

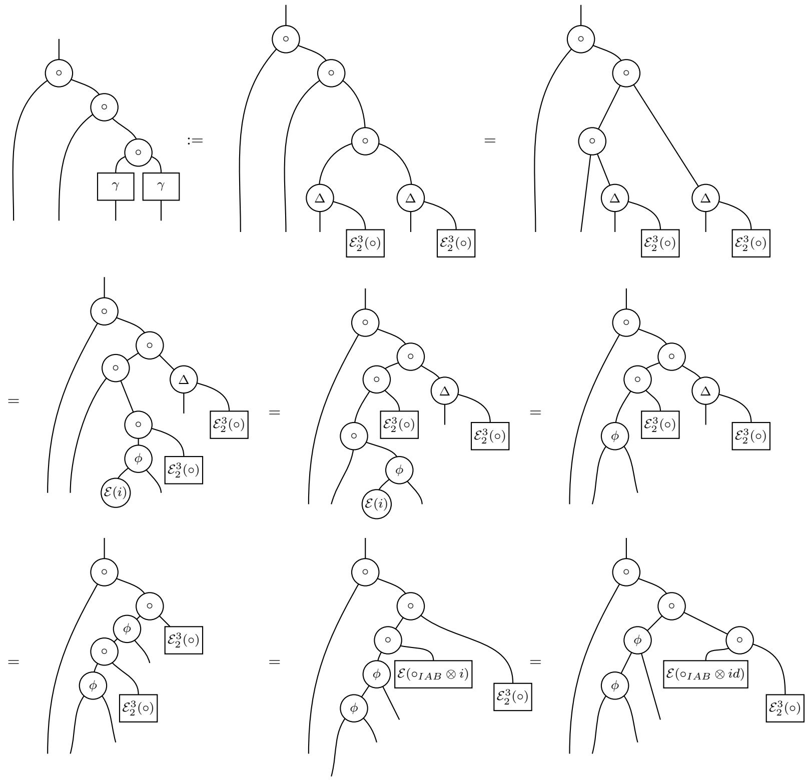
$$\gamma : (\mathcal{C}_{i+2}, \mathcal{C}_i, \bigcirc^s, \overset{s}{\bigotimes}, \mathcal{E}^s, [- \rightsquigarrow^s -]) \longrightarrow (\mathcal{C}_{i+2}, \mathcal{C}_i, \bigcirc^m, \overset{m}{\bigotimes}, \mathcal{E}^m, [- \rightsquigarrow^m -]) \tag{51}$$

defined by the functors $\gamma^C(-) = \mathcal{I}_{\mathcal{C}_i}(-)$ and $\gamma^V(-) = \mathcal{I}_{\mathcal{C}_{i+2}}(-)$ along with the family



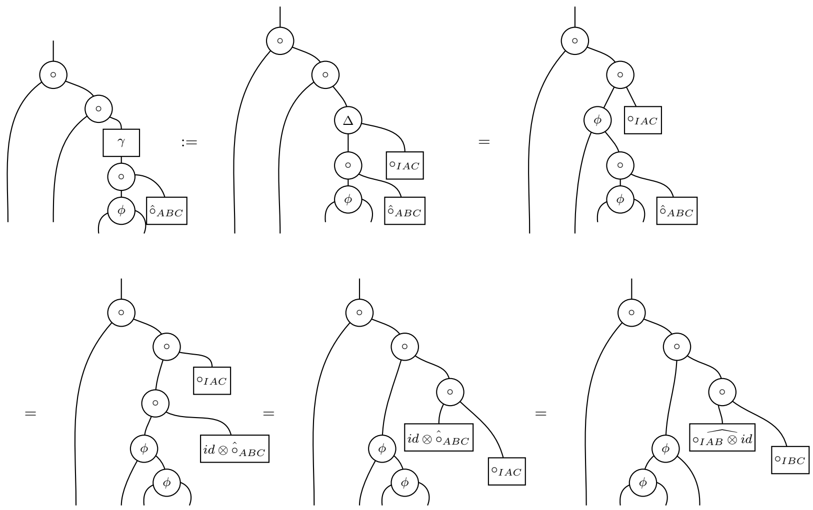
(52)

Proof. We will use from the start the fact that the composition morphisms are completely injective, placing the left hand side of the monoid homorphism condition under two composition morphisms, for readability of the proofs we take all categories $\mathcal{C}_i$ to be strict. The generic non-strict case is proven identically up to tedious additional book-keeping of associators and unitors.



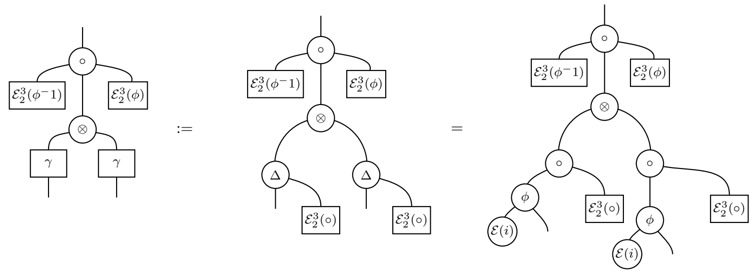
(53)

Then doing the same to the right and side of the condition.



(54)

Next we confirm that the parallel composition preservation condition is satisfied, deriving equality between the following two terms, firstly:



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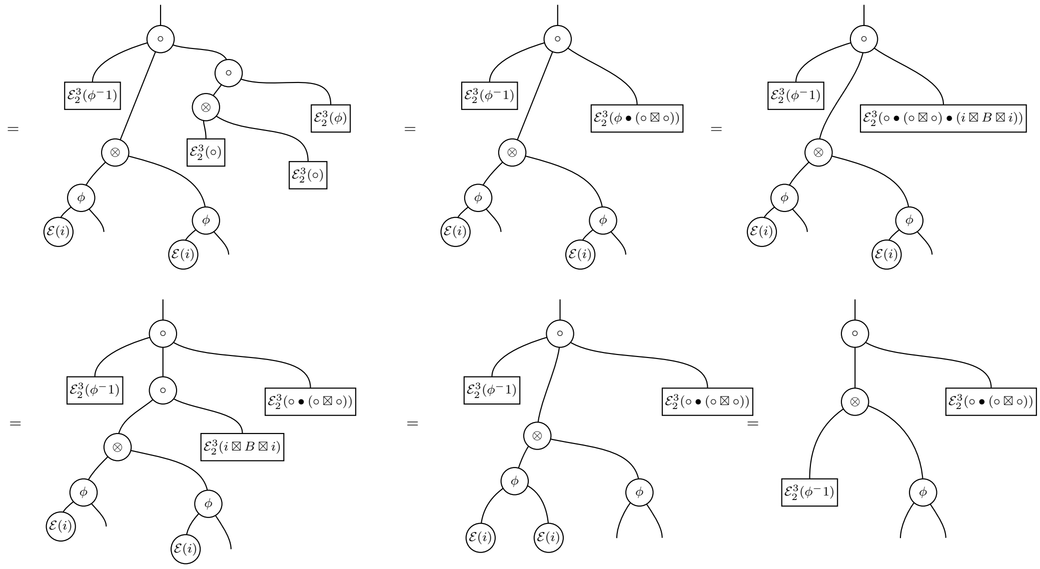

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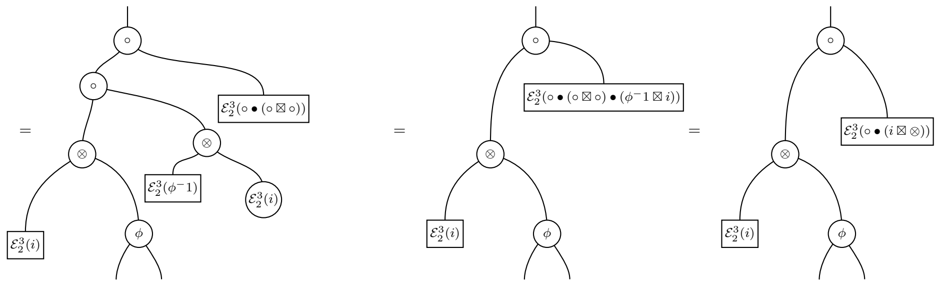
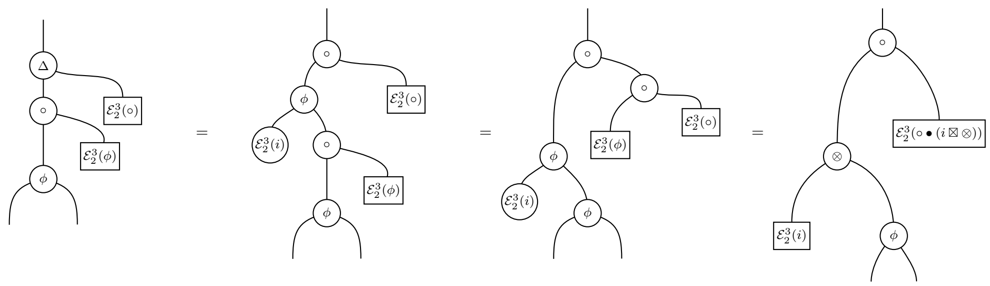
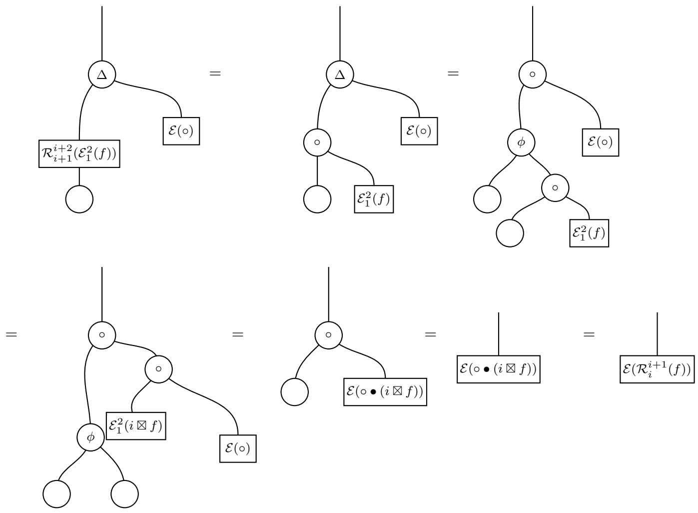


and then secondly:



(57)

Finally the state-based condition can be confirmed:

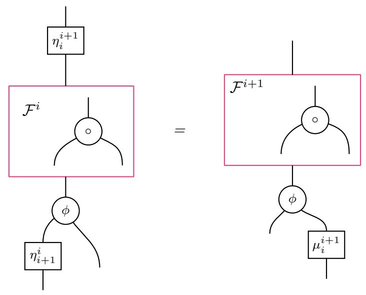


(58)

□

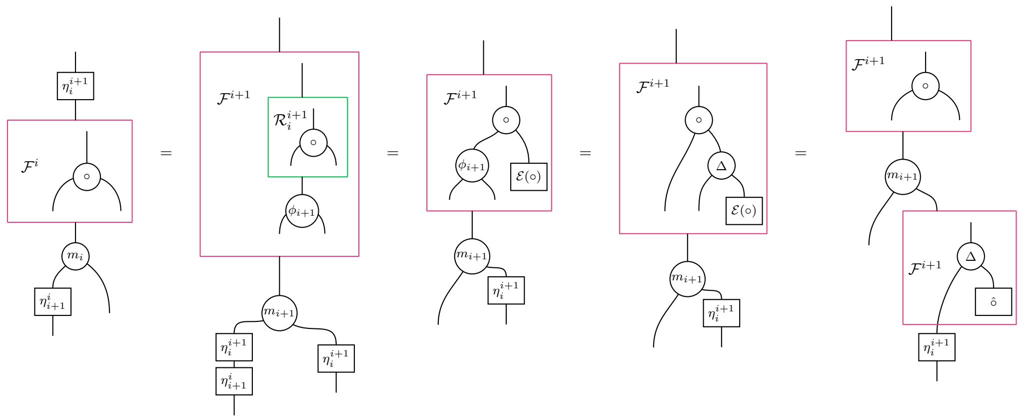
H Proof of Theorem

Lemma 4. *The following condition holds for the isomorphism* $\mu_i^{i+1} := \mathcal{F}^{i+1}(\gamma) \circ \eta_i^{i+1}$



(59)

Proof.



(60)

□

Indeed the above property is the key ingredient in the construction of our main result.

Theorem 15. *The apex* $\mathcal{C}$ *of any Merger of infinite order is a complete super-monoidal category, in other words* $\mathcal{C}$ *is a CSMC such that every* $e_{I \Rightarrow A}$ *is an isomorphism.*

Proof. We work with the following definition of a closed symmetric monoidal category

Definition 19. *An SMC $\mathcal{C}$ is* closed *if for every $A, B \in Ob(\mathcal{C})$ there exists an object $A \Rightarrow B$ and a morphism $\texttt{eval}_{A\Rightarrow B} : A \otimes A \Rightarrow B \to B$, called the* evaluation morphism*, such that for all $f : A \otimes C \to B$ there exists a unique $\bar{f} : C \to (A \Rightarrow B)$ such that $\texttt{eval}_{A\Rightarrow B} \circ (id \otimes \bar{f}) = f$.*

Since the coproduct $\coprod_i \mathcal{F}_i$ is essentially surjective, each object A can be assigned an object X_A an "index" l_A and an isomorphism L_A such that $L_A : A \to \mathcal{F}_{l_A}(X_A)$. A compact notation can be introduced for combinations of functors of the form $[I_i, -]$.

- $\mathcal{R}_i^{i+1} := [I_i, -]$
- $\mathcal{R}_i^j := \mathcal{R}_{j-1}^j \circ \mathcal{R}_i^{j-1}$

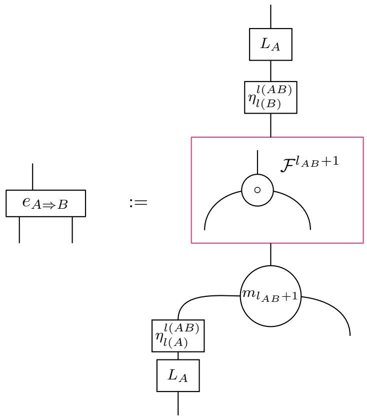
furthermore the function $l : ob(\mathcal{C}) \to \mathbb{N}$ can be extended to lists by

$$l_{AB} := \texttt{max}(l_A, l_A)$$

After which one can define the object representing the space of morphisms from A to B by

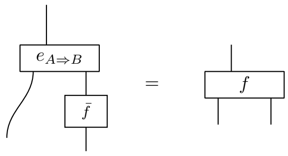
$$A \Rightarrow B := \mathcal{F}_{l_{AB}+1}[\mathcal{R}_{l_A}^{l_{AB}}(X_A), \mathcal{R}_{l_B}^{l_{AB}}(X_B)]$$

This is the object representing the lifting of both A and B in to the $C^{l_{AB}}$ which contains them both, and then using the process object in the next category $C^{l_{AB}+1}$ to represent the processes between them. The characterisation theorem of complete super-monoidal categories as closed monoidal categories with each $e_{I\Rightarrow A}$ an isomorphism, means that one need only provide proof of the latter formulation. For each A,B an evaluation $e_{A\Rightarrow B} : A \otimes (A \Rightarrow B) \to B$ can be defined by,



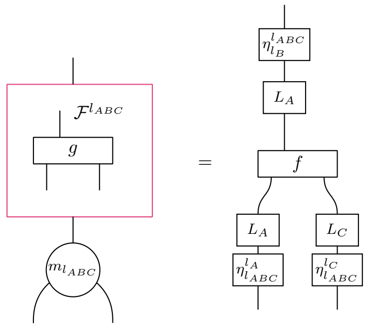
(61)

For $\mathcal{C}$ to be closed monoidal one must show that for every A, B, C and for every $f : A \otimes C \to B$ there exists a unique $\bar{f} : C \to (A \Rightarrow B)$ such that,



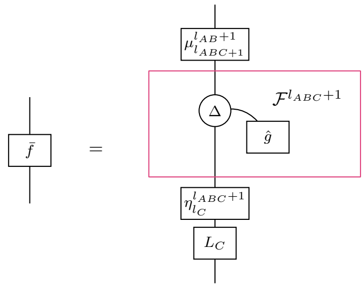
(62)

Indeed such a map $\bar{f}$ can be constructed. Firstly defining g such that



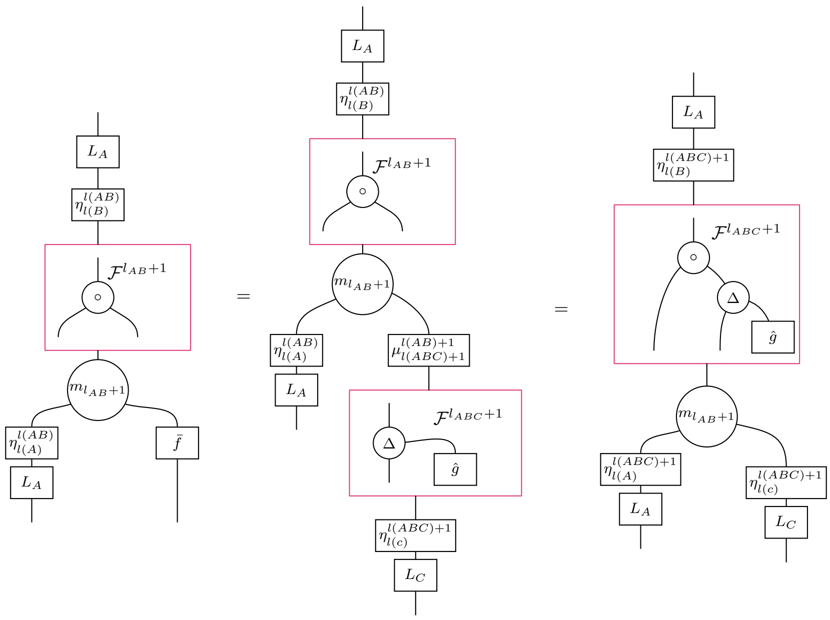
(63)

Such a g must exist since each functor $\mathcal{F}_i$ is full. In terms of this g define $\bar{f}$ by



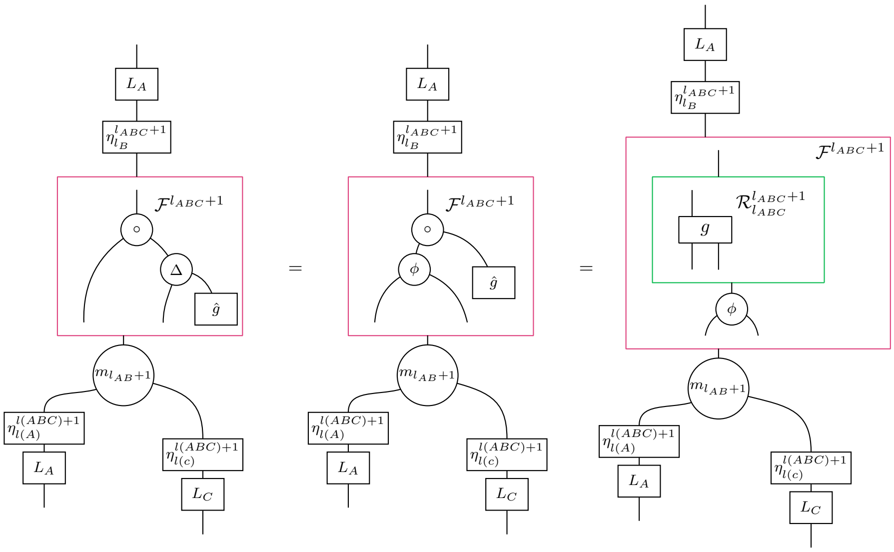
(64)

Then to prove the required identity first requires repeated application of lemma (16),



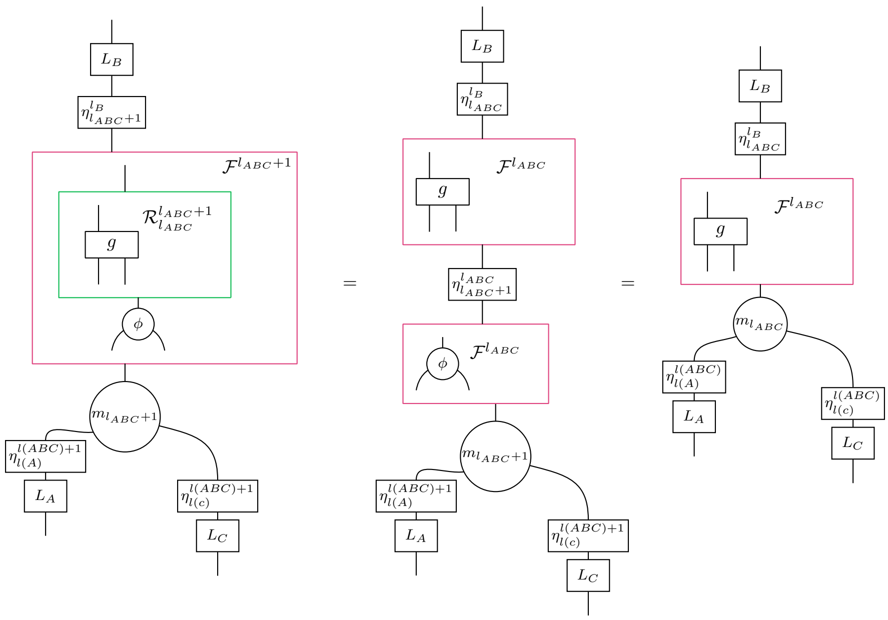
(65)

and then using the defining identity for the partial insertion operation Δ.



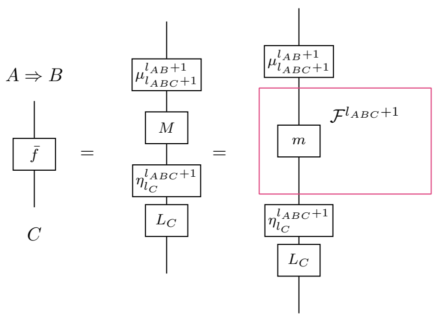
(66)

and finally using monoidal naturality of the transformation $\eta_{l_{ABC}+1}^{l_{ABC}}$.



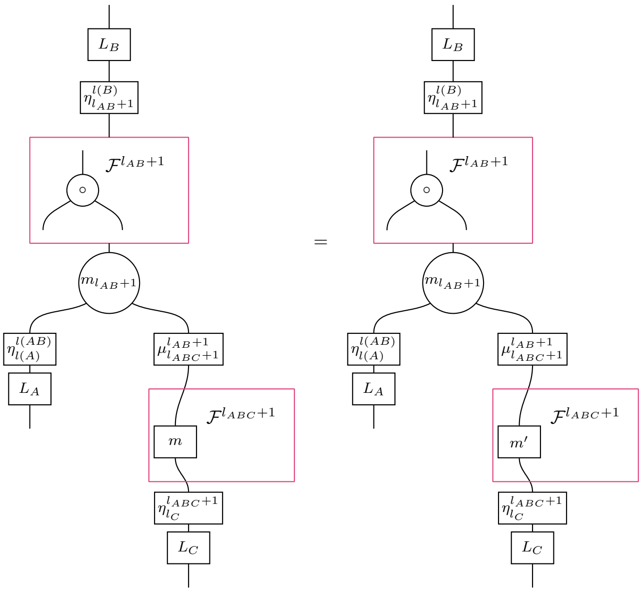
(67)

The morphism $\bar{f}$ satisfying $e \circ (A \otimes \bar{f}) = f$ must be demonstrated to be unique. Every μ_i^j is an isomorphism by coherence of the sequence of super-monoidal categories, as a result every morphism $h : C \to A \Rightarrow B$ can be written in the form



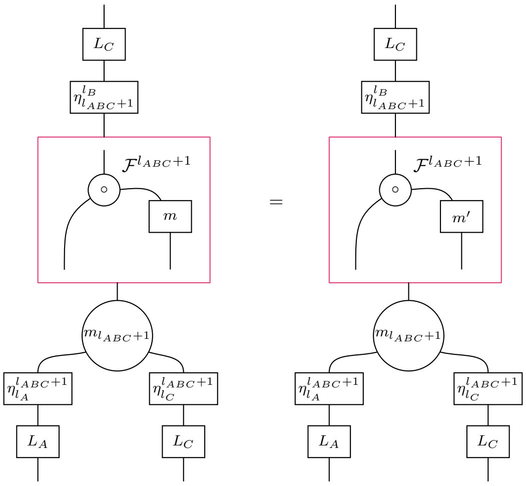
(68)

Where in the last line fullness of each $\mathcal{F}_i$ is used. Assuming h and h with decomposition in terms of m and m' respectively both evaluate to the same morphism $e \circ (A \otimes h) = e \circ (A \otimes h')$:



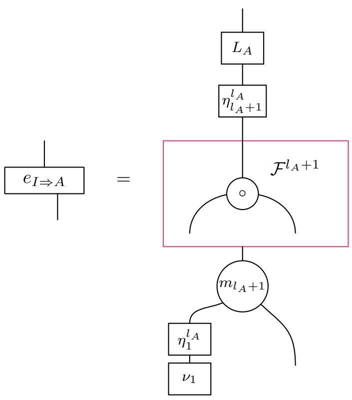
(69)

Which in turn implies



(70)

Since each η and L is an isomorphism, each composition morphism $\bigcirc$ is completely injective, and each $\mathcal{F}_i$ is faithul this entails that $m = m'$ and as a result that $\bar{f} = \bar{f}'$. It follows that $\bar{f}$ is the unique morphism satisfying the evaluation condition for f. The remaining condition to prove is that each $e_{I \Rightarrow A}$ is an isomprphism, firstly each $e_{I \Rightarrow A}$ takes the following form,



(71)

Two properties of the natural transformations η can now be leveraged, firstly a family of commuting squares involving $I_{\mathcal{C}}$ which indeed commute since η is monoidal,

$$\begin{array}{ccccc} & & I_C & & \\ & {\scriptstyle \mathcal{V}_{n-1}} \swarrow & & \searrow {\scriptstyle \mathcal{V}_n} & \\ \mathcal{F}_{n-1}(I_{n-1}) & & & & \mathcal{F}_n(I_n) \\ & {\scriptstyle \eta_{I_{n-1}}} \searrow & & \swarrow {\scriptstyle \mathcal{F}_n(\mathcal{E}^n_{n-1}(i_{I_{n-1}}))} & \\ & & \mathcal{F}_n(\mathcal{R}^n_{n-1}(I_{n-1})) & & \end{array}$$

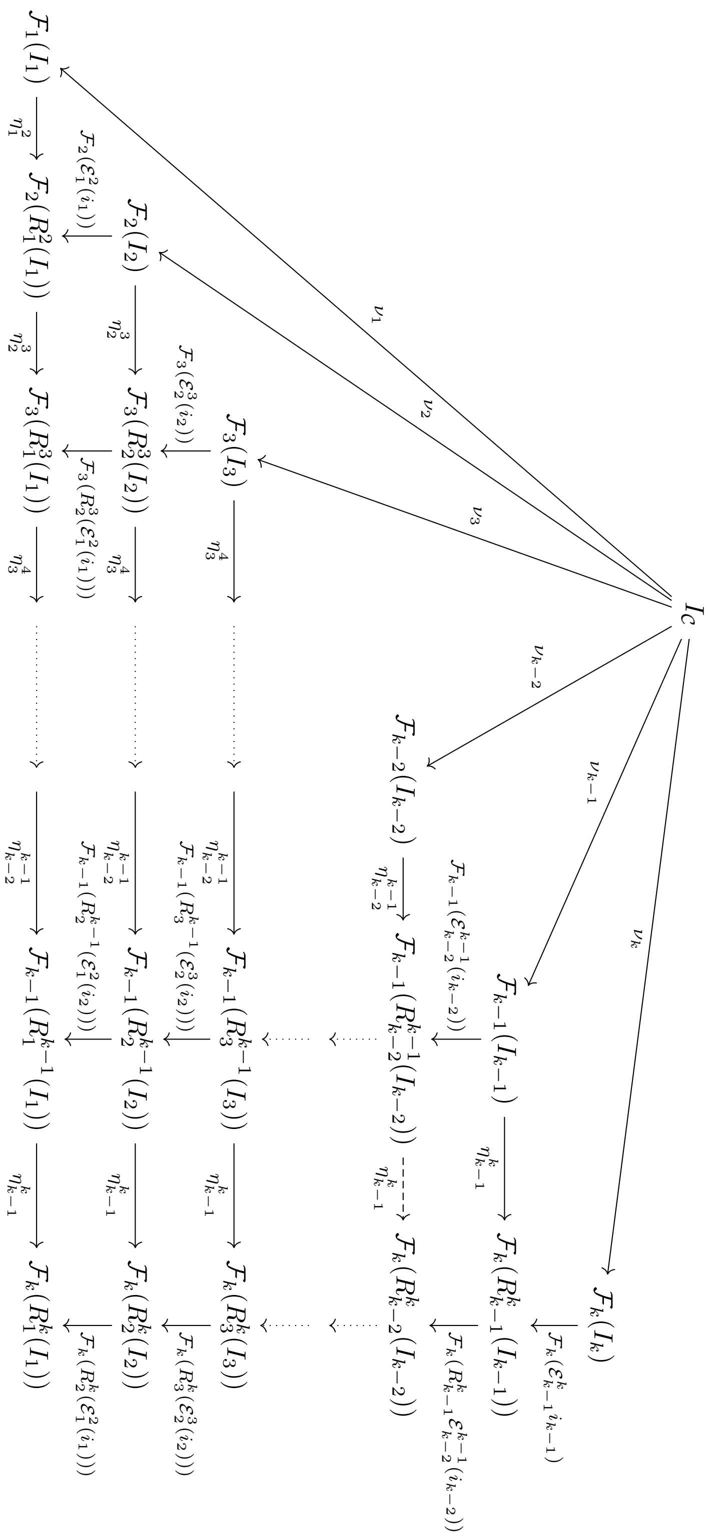
secondly a family of squares which commute since η is natural,

$$\begin{array}{ccc} \mathcal{F}_{n-1}(A) & \xrightarrow{\eta^n_{n-1}} & \mathcal{F}_n(\mathcal{R}^n_{n-1}(A)) \\ {\scriptstyle \mathcal{F}_{n-1}(f)} \downarrow & & \downarrow {\scriptstyle \mathcal{F}_n(\mathcal{R}^n_{n-1}(f))} \\ \mathcal{F}_{n-1}(B) & \xrightarrow[\eta^n_{n-1}]{} & \mathcal{F}_n(\mathcal{R}^n_{n-1}(B)) \end{array}$$

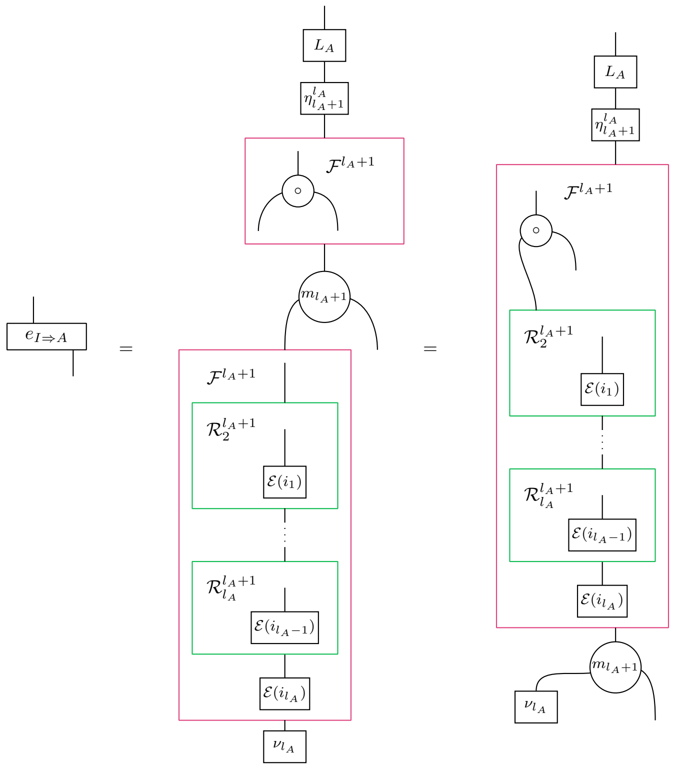
The above two famillies may be used to show that for each k the term $\eta^k_1 \circ \nu_1$ may be replaced by

$$\eta^k_1 \circ \nu_1 = \mathcal{F}_k(R^k_2(i_2)) \circ \mathcal{F}_k(R^k_3(i_3)) \circ \cdots \circ \mathcal{F}_k(i_k) \circ \nu_k$$

using the following commutative diagram built from the above families

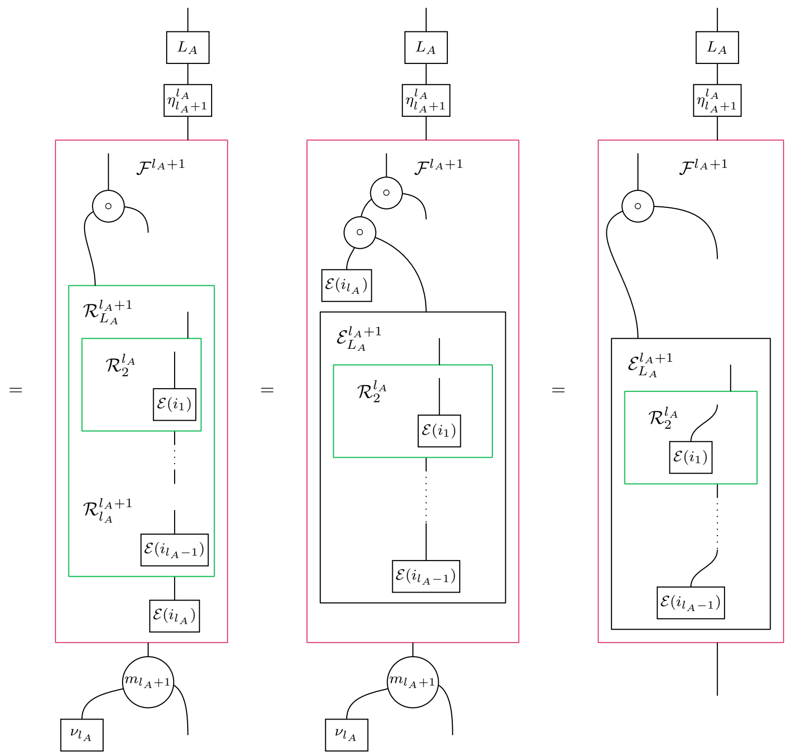


In particular the above identity is given by the two routes along the circumference of this commutative diagram. Using the derived identity the evaluation $e_{I \Rightarrow A}$ is consequently equal the following diagram



(72)

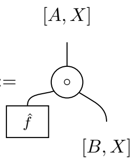
Using the definition of the functor $\mathcal{R}_{l_A}^{l_A+1}$



(73)

The above is a circuit built from the functor $\mathcal{F}_{L_A+1}$ applied to a morphism k along with post composition by an isomorphism. Since the functor $\mathcal{F}_{L_A+1}$ will preserve isomorphism, all that remains is to show that the morphism k on which the functor is applied, is also an isomorphism.

To demonstrate this we note that the mapping

$f \mapsto [f, X] :=$ 

is a contravariant functor, and therefore preserves isomorphism. Furthermore since each raising functor $([I_i, \phi_{i+1}, \mathcal{E}_i^{i+1}(i_{I_i})],)$ is strong monoidal by assumption it follows that each $\mathcal{E}_k^{k+1}(i_k)$ is an isomorphism, each $\mathcal{R}_i^j$ is a functor an so preserves isomorphism, entailing all together that every morphism in the circuit k is an isomorphism.

□